Aspects of M-Branes in M-Theory

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Abstract

This thesis will present work completed throughout the course of my doctoral studies. The thesis primarily concerns developments in the theories of the M5-brane and systems of multiple M-branes in M-theory.

Theories of multiple M2-branes are currently well understood, with the BLG and ABJM models providing a comprehensive and rich base from which to continue study of such objects. The M5-brane however remains shrouded in mystery. A basic multiplet for a single M5-brane is known but scant else directly; it is believed that the six-dimensional $(2,0)$ *model describes the worldvolume theory for multiple M5-branes, however this theory is itself poorly understood. There is therefore much work to be done to gain a deeper understanding of these objects and how they interact. This thesis hopes to be a small part of that story.*

In the first part of this thesis I provide an extended introduction, giving a broad background to many of the topics discussed later in the text. This is intended to set the scene and introduce the major players for what is to follow. I discuss the development of String Theory and its basic results including the spectrum, dualities, and D-branes. Interspersed are diversions into topics which are of relevance for the later work such as the notion of Kaluza-Klein compactification, BPS states, and solitons. I then discuss how String Theory leads inexorably to M-theory and discuss the basics of this new theory. Following this, I dive deeper into a detailed discussion of the BLG theory describing pairs of M2-branes. Next, I move into more recent and tentative models of multiple M-branes which emerge from attempts to extend the $(2,0)$ *algebra to a non-abelian generalisation. This typically involves the addition of non-dynamical form fields into the model which allow for various different brane configuration interpretations.*

I then begin describing the novel systems which constitute my contribution to the field. Firstly, I show how the non-abelian extended $(2,0)$ *theory has a natural sector within it which describes 'null' M2-branes. This system is analysed and is shown to reduce to motion on the moduli space of solutions to the Hitchin system. From this I then find an action for this system which provides a concrete example of a maximally symmetric non-Lorentzian field theory. Another such lagrangian is also found which comes from another extended* $(2,0)$ *system.*

Finally, I discuss a system of M5-branes wrapped on the multi-Taub-NUT geometry. This *system has a natural string theory interpretation of intersecting D4 and D6-branes. Such a brane configuration is known to contain chiral fermions which propagate along the intersection and which arise due to the open strings stretching between the branes. This picture has been previously poorly understood from the M-theory perspective; this work shows the M-theory origin of the chiral states and argues that they are described by a Wess-Zumino-Witten-like model.*

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This work could not have been completed without the constant support and understanding of my wife, Ele, as well as the rest of my family and friends who have now heard far more about strings than they ever bargained for.

Finally, I dedicate this thesis to my late father. He would not have understood what follows, but would have gladly informed me why it was wrong.

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Chapter 1

Introduction

Physicists have long sought to develop a theory with the potential to fully describe all that we see around us in the natural world in a simple, elegant fashion. The Ancient Greeks, attempting to systematise nature into invisible elemental atoms, began the search which leads through Newton and the scientific revolutions of the 18th and 19th Centuries, and into the early 20th Century and Einstein's well known attempts to unify physical laws. This thesis will examine the most promising lead for this age old question: String Theory, or in its modern inception: M-Theory. String Theory began in the 1960s as a model developed to describe hadrons and the strong force more broadly, its genesis was a paper by Veneziano [1] who found a scattering amplitude which was hoped to describe the interactions of these particles. It soon became clear that this model was not likely to be successful for its intended purpose, indeed the strong force would soon after be understood within the framework of Quantum Chromodynamics, however after some work, Veneziano's model was re-understood as a theory of string interactions [2, 3, 4]. Furthermore, the model seemed particularly exciting because of one particular feature—it included a massless spin-two state which could be identified with the graviton, the gauge boson for gravity [5, 6]. This had been a long standing problem in high energy physics—gravity, formulated as a quantum field theory, was known to be non-renormalisable [7] so how could it be united with the other fundamental forces of nature? Principally, string theory provided a solution to this by giving a quantum theory which explicitly and necessarily included a graviton and which, in simple perturbation theory, consists of amplitudes with no UV divergences [8].

Over time string theory developed into a broad, rich theory with many offshoots and has left an enormous footprint on today's landscape of theoretical physics. In particular its aim to provide a theory of everything has been distilled and understood in a modern context within a new idea, that of M-theory, the study of which will constitute the main part of this thesis.

This thesis aims to provide an overview of the subject and report on recent work conducted by myself, in conjunction with my supervisor and other collaborators, over the course of my studies.

The thesis is arranged as follows: In Chapter 1 I begin with a broad introduction to the topic, suitable for a wide audience to follow. This discussion will portray the journey from the idea of strings to its fruition in the modern understanding of M-theory. Along the way, key concepts to be employed later will be reviewed and it is hoped that the reader obtains an appreciation for the scope of the theory and its fundamental ideas. In Chapter 2 a more detailed review of the primary objects of study in this thesis—M-branes—will be conducted. The motivation for their existence and basic properties will be discussed in addition to an overview of our current understanding of their dynamics. The theories discussed in this section will prove essential for later chapters. In Chapter 3 I discuss work which concerns a special case of a system introduced in the preceding chapter and find a novel system of so called "null" M-branes. This work was published in October 2017 in the Journal of High Energy Physics (JHEP) in collaboration with my supervisor, Prof. N. Lambert and colleague Dr. P Kucharski [9]. In Chapter 4 I study two systems: one discussed in Chapter 2 and another from Chapter 3; I show them to be examples of novel, maximally supersymmetric non-Lorentzian theories. This work was published in JHEP in October 2018 in collaboration with my supervisor [10]. In Chapter 5 I change focus and discuss work which looks for the M-theory origin of certain chiral states which are well understood from the string theory perspective. This work was completed in collaboration with my supervisor and was published in JHEP in April 2018 [11]. Finally, in Chapter 6 I provide a conclusion to the work in this thesis.

I proceed now to discuss the background and context for the work in this thesis; I begin with an overview of string theory, followed by a discussion of its development into M-theory with some elementary results presented. For much of the material covered in this chapter I owe a great debt to numerous review articles and books.

In particular the following works provide a comprehensive overview of, and heavily influenced, the fundamentals of String Theory and M-Theory presented here [8, 12, 13, 14, 15, 16].

1.1 String Theory

The basic idea embodied by String Theory is eponymous; we presume that the fundamental matter of the universe is made up of propagating $(1+1)$-dimensional objects, referred to as strings. Initial treatments focused on a bosonic formulation, still seen in introductory textbooks and courses, however this is inadequate on its own due to the presence of tachyonic states in the spectrum of the theory. A selection of references for this initial bosonic treatment are provided here [17, 18, 19, 20, 21, 22, 23, 24]; throughout this section the references provided will aim to give a broad historical overview of the literature and are not intended to be comprehensive. To remove the tachyonic states an extra symmetry was added to the theory, known as supersymmetry, this had the effect of removing the tachyon and also, famously, fixing the number of dimensions in which the theory must live to ten [25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

Strings can come in two different varieties, open and closed, with most theories of strings consisting of closed strings; open strings, as will be seen, generally imply the existence of extra objects in the theory. A spectrum for supersymmetric strings can be identified by looking at the lowest energy excitations and, at a perturbative level, scattering amplitudes can be computed [8]. As in quantum field theory, these amplitudes are governed by some coupling which is small in the perturbative limit but with some strong coupling regime where perturbation theory breaks down. The coupling in String Theory is interesting because it is dynamically determined by the theory. Because of this remarkable fact, one finds that there is only one fundamental parameter in the theory, that of the string length scale, denoted as l_s, which in turn is used to set the tension for the fundamental string, typically one has [15]

$$T = \frac{1}{2\pi l_s^2}. \tag{1.1.1}$$

The string scale is considered very small when compared to our current theories and experiments so massive modes in the theory, whose masses are set by the inverse of the length scale, take a vast amount of energy to excite. This means that, while the massive modes are of course very important to the theory, the massless modes in the theory are the ones of interest for phenomenology and for first forays into the subject. Therefore when giving an overview of the contents of various string theories later, the massless modes will be of primary interest.

From a phenomenological perspective, the dimensional constraint leads to one of string theory's biggest difficulties: the landscape problem. The naive intuition of our merely four-dimensional spacetime naturally contradicts the string theory prediction of ten dimensions and so the question occurs: how should the two be reconciled? The answer appears in the process of compactification; the idea being that the deficit six dimensions form a compact space with small volume so that they are unseen at the scales with which we are experienced [35]. However, at small scales, of order the string scale, these dimensions become consequential. A similar idea had been considered by Kaluza in 1921 [36] and refined by Klein and others into what is now known as Kaluza-Klein theory. The original goal of the construction was to unify electromagnetism with gravity by supposing the existence of a fifth dimension in spacetime. This was ultimately unsuccessful at describing a realistic theory but the methodology and language resulting from this remains a part of the current lexicon. I will briefly discuss the technique here as it will be useful for later.

1.1.1 Kaluza-Klein Theory

This is a rather standard argument but I loosely follow the treatment in [15] here. In the classic formulation one begins with a five-dimensional space of topology $\mathbb{R}^{1,3} \times S^1$. I will consider pure gravity and write down coordinates as x^M, $M, N = 0, 1, 2, 3, 4$ with $x^4 = x^4 + 2\pi R$ periodic; R characterises the radius of the circular S^1 direction. Then, write down the natural action for gravity in five dimensions as

$$S_{5D} = \int \sqrt{-G^{(5)}} R^{(5)} d^5 x. \tag{1.1.2}$$

Here, R is the Ricci scalar and G the determinant of the metric. Throughout, such bracketed indices, as seen above, indicate the dimension in which the object lives—in this case it indicates that the objects are five-dimensional quantities. I will show that after compactifying and reducing on the S^1, this leads to theories which describe gravity unified with electromagnetism.

Perform a split in the indices; $\mu, \nu = 0, 1, 2, 3$ and x^4 separate; one then defines four-dimensional quantities $G^{(4)}$, A_μ, and ϕ such that the field content splits, schematically, to

$$G^{(4)}_{\mu\nu} \sim G^{(5)}_{\mu\nu}, \quad A_\mu \sim G^{(5)}_{\mu 4}, \quad e^{2\phi} \sim G^{(5)}_{44}, \tag{1.1.3}$$

where A_μ is a new $U(1)$ gauge field, and ϕ a scalar known as the dilaton. The idea is then that on scales much larger than R, so that it is small, the theory should look effectively four-dimensional and, in the limit of $R \to 0$, should be independent of x^4; this condition will be looked at more carefully below. One can show that under this choice of periodic coordinate the Ricci scalar reduces to:

$$R^{(5)} = R^{(4)} - 2e^{-\phi}\nabla^2 e^{\phi} - \frac{1}{4}e^{2\phi}F_{\mu\nu}F^{\mu\nu}, \tag{1.1.4}$$

which implies that the action for the four-dimensional theory is given by

$$S_{4D} = \int d^4x\sqrt{-G^{(4)}}\left[R^{(4)} - \partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{2\phi}F_{\mu\nu}F^{\mu\nu}\right]. \tag{1.1.5}$$

Thus the pure gravity theory in five-dimensions has split to four-dimensional gravity, coupled to a $U(1)$ electromagnetic gauge field, with a new scalar known as the dilaton which acts to dynamically set the coupling of the gauge field. Later it will be argued that string theory generates its own coupling in precisely the same way.

Thus Kaluza-Klein theory demonstrates the rich possibilities that come from proposing additional dimensions in the theory. Simple theories in higher dimensions reduce down to more complex theories in lower dimensions. Another way to look at the decomposition which makes clear what happens here is to consider the compactification of one field only.

Take the scalar in five dimensions with full coordinate dependence and fourier expand in the periodic coordinate

$$\phi(x^M) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu)e^{ip_4x^4} = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu)e^{\frac{inx^4}{R}}. \tag{1.1.6}$$

I assume here that the scalar obeys the Klein-Gordon equation and have immediately identified the quantised momentum $p_4 = \frac{n}{R}$ for $n \in \mathbb{Z}$ to enforce the correct periodicity of x^4. Substituting this expansion into the five-dimensional massless

Klein-Gordon equation of motion, $\partial_M \partial^M \phi = 0$ one finds

$$\partial_\mu \partial^\mu \phi_n - \frac{n^2}{R^2} \phi_n = 0. \tag{1.1.7}$$

Thus the compactification on S^1 induces a massless five-dimensional scalar field on the theory which, from the four-dimensional perspective, constitutes an infinite tower of scalar fields labelled by integers n with mass $M_n = \frac{n}{R}$. These are often referred to as Kaluza-Klein (KK) modes and we see that the lowest energy $n = 0$ state corresponds precisely to the massless scalar dilaton found in equation (1.1.5) with the higher mode states becoming heavy for very small R. At small R then such states take large amounts of energy to excite which implies by the usual QFT arguments that they will drop out of the theory at low energy.

Later it will be described how D0-branes arise in string theory from a completely analogous process of dimensional reduction on the M-theory circle; in fact compactifications akin to this arise frequently in the study of M-theory [16]. Furthermore this is the basic idea behind the phenomenological realisation of string and M-Theory. While this compactification is only in one dimension, the hope for string theory is that there is a six-dimensional compact space which, when reduced down from ten-dimensions, can replicate the complexity and chaos of the Standard Model. Such hopes are proving difficult to verify due to the astronomical number of such spaces—possibly as large as 10^{100} [37]—and so efforts have focused primarily on increasing our understanding of the fundamental theory, in the hope that this may shed further light on the phenomenological question, or provide hints for other theories which may one day furnish a description of the world around us.

I now move on to discuss superstring theory in more depth.

1.1.2 Supersymmetric Strings

String theory developed quickly throughout the 80s and 90s under the first and second superstring revolutions. With the addition of supersymmetry, the so called superstring was quickly analysed and its properties derived [32, 33, 34]. The resulting spectrum was found to be tachyon free, due to the GSO projection [31], but after this the analysis became more complicated. Five superstring theories emerged; two, labelled type IIA and IIB come simply from considering a bosonic string with supersymetry imposed and contain in the massless bosonic spectrum a dilaton, a graviton, and a two-form gauge field; collectively known as the NS-NS [38] sector of the theory. The dilaton here is a scalar field, ϕ, which dynamically generates the string theory coupling, $g_s = e^{\phi}$, it will be seen later how this dilaton is in turn generated from a Kaluza-Klein process as described in the previous section. In addition is the R-R [25] sector in which the massless spectrum contains p-form fields with $p = 1, 3$ in the IIA theory and $p = 0, 2, 4$ for the IIB theory. In the type IIB case the four-form has a special property in that its field strength is self dual, this is necessary to match the number of fermionic and bosonic degrees of freedom as supersymetry requires [16]. NS and R stand for 'Neveu-Schwarz' [38] and 'Ramond' [25] respectively and simply indicate different choices of boundary conditions for the fermionic modes on the string. The fermionic content of both theories consists of a pair of dilatinos and a pair of gravitinos leading to maximal $N = 2$ supersymmetry, *i.e.* 32 supercharges. What differentiates the two theories is both the R-R sector, which will be discussed later in the discussion on D-branes, and the chirality of the fermionic sector. In the type IIA theory the gravitinos are of opposite chirality, thus giving a non-chiral $(1, 1)$ theory; whereas in the IIB case the gravitinos have the same chirality so one finds a chiral $(2, 0)$ theory. The bulk type II string theories are theories of closed strings however it will be seen later that open strings do exist in the theory with their end points constrained to end on new objects known as D-branes.

The remaining string theories will be less important for the remainder of this thesis but I will discuss them briefly here to complete the picture and to motivate the shift later into M-theory. The type I theory is best understood as arising from a projection of type IIB onto states invariant under world sheet parity [16]. This transformation maps the spatial string world sheet coordinate, σ as

$$\Omega : \sigma \to -\sigma. \tag{1.1.8}$$

Such a transformation swaps the left and right moving modes around the string, and so is a symmetry of type IIB but not type IIA since only in the IIB case do the left and right moving fermions carry the same chirality. One projects onto a particular choice, $\Omega = +1$ say, and this has the result of projecting out half of the states, in particular one of the gravitinos disappears. The resultant type I theory now has $N = 1$ supersymmetry and a coupled $SO(32)$ Yang-Mills theory which is necessary for appropriate anomaly cancellation in the theory. The type IIA, IIB, and I string theories were all outlined, and their spectrum described by Green and Schwarz in [32, 33, 34].

The final two string theories discovered are known as the Heterotic theories [39, 40, 35] and are obtained by a slightly odd prescription of enforcing supersymmetry in the left moving string modes while not enforcing such supersymmetry in the right moving modes. This unorthodox setup actually leads to two interesting $N = 1$ supersymmetric theories of closed strings. The content of such theories is extremely attractive phenomenologically speaking as they contain non-abelian gauge symmetry and chiral fermions; both key components necessary in the standard model. The gauge symmetries distinguish the two theories with one having an $SO(32)$ gauge symmetry and the other $E_8 \times E_8$. Given their similarity with features found in the standard model the two heterotic theories were the subject of much focus soon after their discovery, eclipsing the other theories; it was later on, after certain dualities had been discovered, that all theories started to look equally interesting.

These disparate theories, it is now believed, are all unified by the encompassing M-theory as will be discussed below. However, the story is not quite finished, at least for the type II theories as previously alluded to. The form fields found in the R-R sector give rise to new dynamical objects in the theory known as D-branes. It is with the intention of understanding this phenomenon that I now proceed.

1.1.3 Dualities in String Theory

With five kinds of string theory categorised, the natural question is whether they are related to one another at all. This section will discuss the dualities which unite the theories. Some time and effort will be spent on T-duality as it will lead to objects highly relevant for this thesis, while other dualities will be mentioned more briefly in Section 1.2.

T-Duality

The first and most instructive duality for the purposes of this thesis is T-duality. This was first discussed explicitly in [41] and developed in [42, 43]. Reviews are commonplace and can be found in any of the textbooks listed earlier, in particular, [15] and [16]. Both of these have inspired the following treatment.

The idea of T-duality is to consider string theory where we compactify one dimension onto a circle. I will provide here the argument for the bosonic modes of the closed type II string only, leaving the fermions to one side. I take $X^9(\tau, \sigma)$ to be the spacetime coordinate compactified on a circle of radius R so that one should have periodic boundary conditions for this coordinate

$$X^9(\tau, \sigma + \pi) = X^9(\tau, \sigma) + 2\pi R w. \tag{1.1.9}$$

Here I take the circumference of the string to be π, and allow for the possibility that the closed string wraps the dimension multiple times, with w being this winding number.

The appropriate expansion for this coordinate, which respects the boundary conditions is then

$$\begin{aligned} X^9(\tau, \sigma) &= x^9 + 2l_s^2 p^9 \tau + 2Rw\sigma + \ldots \\ &= x^9 + 2l_s^2 \frac{n}{R}\tau + 2Rw\sigma + \ldots . \end{aligned} \tag{1.1.10}$$

Here, x^9 is the centre of mass position of the string, p^9 the corresponding momentum and the ellipsis hides the higher oscillatory modes. In addition I have identified the quantised momentum in the second equality using the same argument as in the Kaluza-Klein setup discussed in Section 1.1.1. One can now split this expression into left and right moving modes, $X^9_L(\tau + \sigma)$ and $X^9_R(\tau - \sigma)$

$$\begin{aligned} X^9_R(\tau - \sigma) =& \frac{1}{2}(x^9 - \tilde{x}^9) + (l_s^2 \frac{n}{R} - wR)(\tau - \sigma) + \ldots \\ X^9_L(\tau + \sigma) =& \frac{1}{2}(x^9 + \tilde{x}^9) + (l_s^2 \frac{n}{R} + wR)(\tau + \sigma) + \ldots \end{aligned} \tag{1.1.11}$$

where $\tilde{x}^9$ is a dummy variable for now, the interpretation of which will become clear below.

This allows, with certain other constraints on the Virasoro generators, a derivation for an expression of the mass of states in such a situation:

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{l_s^4} + \ldots \tag{1.1.12}$$

The ellipsis in this case indicates terms which measure the excitation of left and right moving modes respectively—these are the terms that allowed for the analysis of the spectrum seen in Section 1.1.2. Of import here though is the behaviour of the first two terms; referred to as the KK modes and the winding modes respectively. One sees that a few curious features emerge. As $R \to \infty$ the winding modes get progressively heavier; consequently less accessible, and thus drop out of the theory. Conversely, the gaps between the KK tower of states get progressively thinner so as to approach a continuum; in the limit $R \to \infty$ then, as intuition suggests, the compactified dimension unravels and one finds the usual uncompactified theory.

Now, as $R \to 0$ the usual KK analysis would follow Section 1.1.1 where the second term is not present and the KK modes become extremely heavy so drop out the theory entirely and we have a full dimensional reduction. However, the stringy aspects of the theory are paramount here and the winding modes are included. Thus as $R \to 0$ the winding modes approach a continuum as it is very easy to excite them around a shrinking circle. This then implies the re-emergence of a large uncompactified dimension. To state this another way, in either case, when R goes to zero or infinity, a large uncompactified dimension emerges; this is initially very unintuitive.

Symmetry of equation (1.1.12) shows the origin of this; under a transformation $n \leftrightarrow w$ and simultaneous $R \to \tilde{R} \equiv \frac{l_s^2}{R}$ it is seen that (1.1.12) is invariant. Thus, this T-duality is relating a theory compactified on a circle of radius R to another theory compactified on radius $\sim \frac{1}{R}$.

To relate this back to the type II string theories I first refer back to the left and right moving mode expansion from (1.1.11) and see that the symmetry transformation described above is equivalent to the transformation

$$X_L \to X_L, \qquad X_R \to -X_R \tag{1.1.13}$$

where under T-duality the x^9 coordinate is now considered dummy and the $\tilde{x}^9$ is the centre of mass position. In other words

$$\begin{aligned}\tilde{X}^9(\tau, \sigma) =& X_L^9(\tau\sigma) - X_R^9(\tau\sigma) \\ =& \quad \tilde{x}^9 + 2l_s^2 \frac{n}{R}\sigma + 2Rw\tau + \dots .\end{aligned} \tag{1.1.14}$$

It is clear that in this case the KK modes wrapping the T-dualised circle, comparing to (1.1.10), have conjugate momentum given by

$$p^9 = \frac{Rw}{l_s^2} = \frac{w}{\tilde{R}}. \tag{1.1.15}$$

And thus one sees that T-duality really is relating a theory with KK momentum running round a circle of radius R with winding modes to a theory with momentum now characterised by the original winding modes, running round a dual circle of radius $1/R$, and winding now characterised by the original KK integer n. This is how T-duality manifests itself for closed strings, how it behaves with open strings will lead to interesting results.

Open Strings

I now provide a brief discussion of how T-duality affects open strings in type II string theory. Even though such strings do not exist in the bulk spacetime of the theory this analysis will provide the bedrock for the discussion of the following section concerning the conditions under which open strings can be found. The argument follows much of that in [16].

The natural action for a free bosonic string is known as, in one form, the Nambu-Goto action [19] and is constructed by analogy with a particle where the action is taken to be proportional to a particle's geometric worldline. In another formulation this action is known as the Polyakov action [44, 24]. Taking this action, gauge fixing it and then varying yields:

$$\begin{aligned} S &= -\frac{1}{2\pi l_s^2}\int d\tau d\sigma \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \\ \delta S &= -\frac{1}{\pi l_s^2}\int d\tau \partial_\sigma X_\mu \delta X^\mu|_{\sigma=0}^{\sigma=\pi}. \end{aligned} \tag{1.1.16}$$

Here, α, β are the worldsheet coordinates τ, σ; and μ, ν are ten-dimensional spacetime coordinates characterising the embedding of the string into the bulk.

For an open string one can impose two kinds of boundary condition, Neumann and Dirichlet. Neumann boundary conditions are those where $\partial_\sigma X_\mu|_{\sigma=0}^{\sigma=\pi} = 0$ is enforced on each coordinate, and Dirichlet are those where $\delta X^\mu|_{\sigma=0}^{\sigma=\pi} = 0$ is imposed on each coordinate. In order to maintain Lorentz invariance, typically one would pick the Neumann conditions for each coordinate and leave it at that. Thus the perturbative spectrum of type II string theory contains only closed strings generically [8]. T-duality will be shown to change this analysis.

The setup is as follows, place an open string with Neumann boundary conditions on all coordinates in the theory where X^9 is compactified to a circle so that it obeys appropriate periodicity conditions as seen in Section 1.1.3. Then solve for this coordinate from the action above, impose the boundary conditions and split into right and left moving modes to find an expansion for the coordinate as

$$\begin{aligned} X^9_R(\tau-\sigma) =& \frac{1}{2}(x^9-\tilde{x}^9) + l_s^2 p^9(\tau-\sigma) + \frac{il_s}{2}\sum \frac{1}{n}\alpha_n e^{-in(\tau-\sigma)} \\ X^9_L(\tau+\sigma) =& \frac{1}{2}(x^9+\tilde{x}^9) + l_s^2 p^9(\tau+\sigma) + \frac{il_s}{2}\sum \frac{1}{n}\alpha_n e^{-in(\tau+\sigma)} \end{aligned} \tag{1.1.17}$$

where the oscillatory modes have now been left in.

If one sums these, the linear sigma terms cancel, the exponentials form a cosine function, $\cos n\sigma$ and thus, since this cosine contains all the remaining σ dependence, one sees that the Neumann boundary conditions are currently enforced—*i.e.* $\partial_\sigma \cos(n\sigma)|_{\sigma=0}^{\sigma=\pi} = 0$. Now T-dualise; the dual coordinate is $\tilde{X}^9 = X^9_L - X^9_R$ and one sees that the exponentials now combine to form $\sin n\sigma$. This corresponds to a shift in the boundary conditions obeyed by the coordinate—the T-dual direction now has fixed, hence Dirichlet, boundary conditions.

Explicitly, one can calculate the boundary conditions obtained to be

$$\tilde{X}^9(\tau,0) = \tilde{x}^9, \qquad \tilde{X}^9(\tau,\pi) = \tilde{x}^9 + 2l_s^2\frac{n}{R}\pi = \tilde{x}^9 + 2n\tilde{R}\pi \tag{1.1.18}$$

where the momentum has been quantised and the duality relationship utilised. Such expressions show how an open string can wrap around the compact dimension n times in a topologically stable way since the string cannot unwrap itself given fixed end points [15].

One also sees that the reverse to that presented here would also be true; one could T-dualise a coordinate initially with Dirichlet boundary conditions to obtain Neumann boundary conditions on the dual circle. Thus a T-duality transformation on an open string simply switches the boundary conditions imposed on that direction. There is more to say on the T-duality of open strings but that will be left for Section, 1.1.4.

One of the key results from this T-duality found in the type II theories comes from the behaviour of X_L and X_R under the transformation. Since the right moving modes change sign under the transformation and since type IIA string theory is a non-chiral theory with opposite chirality gravitinos, one sees that the transformation acts to switch the chirality of one gravitino and thus the theory becomes a chiral theory exactly like type IIB string theory. The converse is also clearly true. Up to the R-R form fields then it seems like T-duality transformations relate type IIA and IIB string theories together—they are one theory viewed in a different way. The discussion of how the forms are reconciled will await the next section but for now it will suffice to simply state that T-duality really is a valid duality relationship which can be shown to hold order by order in perturbation theory [45].

Finally, before leaving the discussion about T-duality for a time, I remark that a similar result holds between the two heterotic theories [46]—implying that these two are simply different aspects of the same theory. The details for how this comes about are different as in the type II case and will not be of concern for this work.

1.1.4 D-branes

As discussed above, both the type IIA and IIB string theories contain various p-form gauge fields from the R-R sector of the massless field content; for IIA, $p = 1, 3$, for IIB, $p = 0, 2, 4$. T-duality has suggested that these two theories are one and the same however I have yet to show how these forms relate to one another.

To see the significance of these forms first consider the usual electromagnetic theory, described by a one-form $A = A_\mu dx^\mu$. Such a field naturally couples to a manifold by the action

$$S = e\int A = e\int dt A_\mu \frac{dX^\mu}{dt} \tag{1.1.19}$$

where one considers the manifold to be a point particle of electric charge e, moving only in a time direction. All this describes is the pull back of the bulk gauge field, A, onto the worldline of a particle. Such a worldline is referred to as a zero-brane where zero indicates the number of spatial dimensions associated to the manifold. The natural generalisation to $(p+1)$-forms coupling to p-branes is given by the corresponding pullback of bulk coordinates X onto the brane parametrised by σ as

$$S = \mu_p \int A = \mu_p \int d^{p+1}\sigma A_{\mu_1\ldots\mu_{p+1}} \varepsilon^{\sigma_1\ldots\sigma_{p+1}} \frac{\partial X^{\mu_1}}{\partial\sigma_1}\cdots\frac{\partial X^{\mu_{p+1}}}{\partial\sigma_{p+1}} \tag{1.1.20}$$

for some electric-like charge μ_p and with ε denoting the totally antisymmetric tensor, defined by $\varepsilon^{01\cdots} = +1$.

The take away is that $(p+1)$-form gauge fields naturally couple to $(p+1)$-dimensional manifolds labelled as p-branes. Such branes have a natural charge associated to them; considering the electromagnetism example once more, the electric and magnetic charge on a particle can be computed, from the field strength $F = dA$, and corresponding hodge dual, $\star F = \frac{1}{2}\varepsilon^{\mu\nu}{}_{\rho\sigma}F_{\mu\nu}dx^\rho dx^\sigma$ as

$$e = \int_{S^2} \star F \qquad g = \int_{S^2} F \tag{1.1.21}$$

where one integrates over a space enclosing the point particle, *i.e* the manifold to which the gauge field is naturally associated; in this case, a 2-sphere. g should be thought of as the magnetic charge associated to a point like magnetic source.

These come from the integral over all space of Maxwell's equations with sources

$$dF = \star J_m \qquad d \star F = \star J_e \tag{1.1.22}$$

and by using Stokes' theorem. The current one-forms are defined by the charge and current densities, $J = (\rho, \vec{j})$, and the existence of magnetic monopoles is assumed by the existence of a corresponding J_m.

The final point to note, which I will not derive here, is the Dirac quantisation condition [47] on the electric and magnetic charges which constrains one to the other as

$$eg \in 2\pi\mathbb{Z}. \tag{1.1.23}$$

Such a constraint follows naturally from gauge invariance on systems of electric charges and magnetic monopoles interacting.

p-branes are a simple extension of this model to sources of higher dimension. One takes the associated form-field, A_{p+1}, finds the corresponding field strength, F_{p+2}, computes the hodge dual to this and thus obtains values for the charges. Note that the field strength is a $(p+2)$-form and hodge dual is given by a $(10-p-2) = (8-p)$-form. This assumes a bulk ten dimensional theory. These give the sphere dimensions over which one should integrate to find the respective charges; geometrically these are the particular n-spheres which can surround a p-brane in ten dimensions.

$$\mu_p = \int_{S^{8-p}} \star F_{p+2} \qquad \mu_{6-p} = \int_{S^{p+2}} F_{p+2}. \tag{1.1.24}$$

The labelling of the second charge may seem odd here; the hodge dual operation is in fact more meaningful than a simple mathematical trick. If one imagines the hodge dual form as being the field strength of a new form-field, $dB_{7-p} = \star F_{p+2}$ then this new field should be interpreted as being associated to another brane object, in this case a $(6-p)$-brane. This is the origin of the labelling convention above.

These charges also obey a generalised Dirac quantisation condition [48, 49, 50], so that one has

$$\mu_p \mu_{6-p} \in 2\pi\mathbb{Z}. \tag{1.1.25}$$

Certain systems or objects can exhibit multiple charges, such constructions are typically interpreted to be bound states of the individual p-branes.

Thus, given a theory with a $(p+1)$-form field the existence of two additional objects in the theory should be inferred, one being a p-brane, and the other its magnetic dual, a $(6-p)$-brane [15].

Turning back to the type II string theories, the various R-R sector form fields present in the spectrum, using the arguments described above, imply the existence of extended branes in the theory known as Dp-branes. Such branes had first been seen in [51, 52] but the key paper linking D-branes to the R-R charges was from Polchinski [53]. In type IIA string theory, with the one-form and three-form R-R fields, one should expect the presence of D0, D2, D4, and D6-branes. Similarly, in the IIB theory, given the zero-form, two-form, and four-form R-R fields, one should expect the presence of D1, D3, D5, and D7-branes as well as naively a D(-1)-brane due to the zero-form field. This object should be interpreted as a D-instanton [54], an object which is totally localised in both space and time. It is worth noting that in the IIB theory the four-form, as mentioned earlier, has a self dual field strength—thus the D3 brane is both electrically and magnetically charged under this field. Finally, the fundamental string can be re-interpreted using similar arguments [16]. The two-form NS-NS sector field found in both type II string theories should be considered as giving rise to the fundamental string. Additionally, there is a corresponding magnetic dual to this known as the NS5-brane. All these objects have tensions which can be computed to be [55]

$$T_{F1} = \frac{2\pi}{(2\pi l_s)^2} \qquad T_{NS5} = \frac{2\pi}{g_s^2(2\pi l_s)^6} \qquad T_{Dp} = \frac{2\pi}{g_s(2\pi l_s)^{p+1}}. \tag{1.1.26}$$

Notice that the D-branes are inherently non-perturbative objects in string theory given their inverse relationship with the string coupling—they are very heavy and thus largely rigid at weak coupling, becoming light only in the strong coupling limit. This is why, when considering only the massless modes and perturbation theory, there is no trace of these objects in the theory.

One might then ask how do D-branes behave under T-duality transformations. The answer is precisely as described in the discussion of Section 1.1.3. It was described there that under T-duality transformations open strings change their boundary conditions along the coordinate which is T-dualised and Dirichlet boundary conditions describe open strings whose end points are fixed along some direction. This is the alternative motivation for these D(irichlet)-branes; one should think of D-branes as being objects on which open strings end—D-branes are precisely the fixed point of the boundary conditions [16]. So how do such objects behave under T-duality? If one takes a D4-brane for example, and T-dualises along a direction parallel to the brane, then this parallel direction was previously a free direction of motion for the open string, as the open string is fixed to the brane but can move along it. The T-duality now fixes the position of the string along this direction and so the string has one less coordinate to move in thus the D4-brane has effectively become a D3-brane with a dual circle transverse to the brane. If one T-dualises the original D4 brane in a direction transverse to the brane then the open string unlocks a previously fixed direction of motion on this new dual circle and the D4-brane becomes a D5-brane. Thus one sees that under a T-duality transformation all odd Dp-branes become even, and all even become odd. Hence T-duality really does transform type IIA string theory into type IIB as was asserted earlier [45].

What should not be unappreciated in this is that D-branes are dynamical objects in their own right. For example they interact and couple to bulk fields like the graviton and form fields, as described above, and so have tensions and charges [15]. They are objects existing in the bulk which allow open strings to stretch between them. Due to the Dirichlet boundary conditions imposed on these objects they clearly break Lorentz invariance in the string theory vacuum to $SO(1,9) \to SO(1,p) \times SO(9-p)$, *i.e* Lorentz invariance on the brane and rotational symmetry in the space transverse to the brane. Supersymmetry of the bulk theory is also broken [15]. One way to see this is that closed type II strings have two gravitinos which generate the supersymmetry, one can think of these as being left moving and right moving currents around the closed string which add to 32 supercharges. For an open string there are not two distinct modes on the string, only one. On a classical string this is because the single mode is reflected off each end-point and hence travels back and forth. Thus the open string carries only one gravitino and the D-branes conserve half the supersymmetry of the bulk [56]. This makes D-branes, and later M-branes, 1/2 BPS states of the theory (often just referred to as BPS states). This is often seen as conditions imposed on the parameters of supersymmetry transformations like $\Gamma_{01\ldots p}\epsilon = \epsilon$. Being a BPS state is extremely valuable for these objects as quantities associated with such objects, for example the tension of the brane, are fixed to not vary for all couplings in the theory by supersymmetry [57]. That is, one can trust such results for generic coupling in the theory. In particular therefore the tension can be calculated at weak coupling and one should not expect any corrections to be needed when extrapolating to strong coupling.

The concept of BPS states recurs throughout string theory and will do so in this thesis so I take a moment here to clarify what is meant. This discussion will lead on to another related topic—solitons—which I will also overview before returning to the main story.

BPS States

BPS states were first discovered classically as lower bounds for the energy of Yang-Mills magnetic monopole solutions, thus they are vacuum conditions on these theories. Much of this discussion is inspired by that in [58]. The classic case comes from the Georgi-Glashow model [59], an $SO(3)$ gauge theory coupled to a Higgs field described by the lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{1}{2}D_\mu\phi_a D^\mu\phi^a - V(\phi). \tag{1.1.27}$$

Lower latin indices represent the $SO(3)$ gauge indices, and fields are in the adjoint representation. The potential is the standard Higgs potential

$$V(\phi) = \frac{1}{4}\lambda(\phi^2 - a^2)^2. \tag{1.1.28}$$

Such a model has electric and magnetic fields given by $E_i = F_{0i}$, and $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$ as well as a conjugate momentum for the Higgs field $\Pi_a = D_0\phi_a$. The theory then admits an energy density function of the form

$$\mathcal{H} = \frac{1}{2}E_i^aE_i^a + \frac{1}{2}B_i^aB_i^a + \frac{1}{2}\Pi^a\Pi^a + \frac{1}{2}D_i\phi^aD_i\phi^a + V(\phi). \tag{1.1.29}$$

The system presented here describes the t'Hooft Polyakov monopole [60, 61], a set of smooth static topological magnetic monopoles. The Bogomol'nyi-Prasad-Sommerfield (BPS) bound [62, 63] describes the smallest possible mass for such a monopole. It is calculated using a standard trick where one recognises that all terms in the energy density function are positive-semi-definite and constructs an inequality accordingly. I work in the centre of mass frame so all the energy is centred in the mass. A bound can be placed on the mass initially by dropping the momentum and potential terms and thus observing

$$\begin{aligned} M \geq& \frac{1}{2}\int_{\mathbb{R}^3} E_i^2 + B_i^2 + (D_i\phi)^2 \\ =& \frac{1}{2}\int_{\mathbb{R}^3} (E_i - D_i\phi\sin\theta)^2 + (B_i - D_i\phi\cos\theta)^2 + 2\sin\theta E_i^aD_i\phi_a + 2\cos\theta B_i^aD_i\phi_a \\ \geq& \int_{\mathbb{R}^3} \sin\theta E_i^aD_i\phi_a + \cos\theta B_i^aD_i\phi_a \end{aligned} \tag{1.1.30}$$

Where an angular parameter, θ was introduced and positive definite terms were dropped in the final inequality. The last two terms can then be written as total derivatives by the Bianchi identity and hence contribute only a surface term at infinity. However, this surface term gives precisely the electric and magnetic charges of the monopole, given in (1.1.24). In addition, since I integrate over the sphere at infinity, the Higgs field should be in its vacuum state, so $\phi \to a$. Thus a BPS bound is obtained

$$M \geq ag\cos\theta + aq\sin\theta. \tag{1.1.31}$$

This bound is maximised when the right hand side is maximised which implies $\tan\theta = \frac{q}{g}$ which implies the stricter BPS bound

$$M \geq a\sqrt{q^2 + g^2}. \tag{1.1.32}$$

To saturate this bound, the condition to define classical BPS states, one should first ensure that the initial terms dropped are set to zero. This implies static solutions where the timelike component of the gauge field is zero and there is no time dependence so that $D_0 = 0$. The static condition implies that $E_i = 0$; thus $\sin\theta = 0$. Next one demands that the positive semi-definite terms in (1.1.30) are zero which leads to the Bogomol'nyi equation [62]

$$B_i = \pm D_i \phi \tag{1.1.33}$$

with $\pm$ allowing for either solution to $\sin\theta = 0$. One final feature is that in order to guarantee the potential vanishing over all space whilst maintaining a non-zero magnetic charge the constant λ must be zero [63].

Further analysis of this theory implies that the gauge boson has mass $M_g = aq$ with q units of electric charge only, while the monopoles have $M_m = ag$ with g units of magnetic charge only [58]. This observation will be useful later when discussing S-duality.

Thus far I have considered only a non-supersymmetric theory and found the existence of a bound on the model. I will not review here the construction of supersymmetry, though it will be extremely important for the remainder of this thesis. Introductions can be found in [64, 65]. In [57] an $N = 2$ supersymmetric version of the above described theory was constructed which has the effect of finding an equivalent BPS condition in such a theory. Such a theory can also be found from dimensional reduction of six-dimensional $N = 1$ supersymmetric Yang-Mills [66]. The BPS bound found in these extended supersymmetry theories can be shown to be totally equivalent to that presented above and arises from a similar argument [67, 57]. In general, theories with extended supersymmetry contain a notion of a BPS state that arises algebraically from the supersymmetry algebra, I review this construction briefly below. In an extended supersymmetry multiplet the algebra takes the following schematic form [65]:

$$\{Q, Q\} = 2Q^2 \sim M + Z \tag{1.1.34}$$

where the Z are the central charges and M is the mass, as I work in the centre of mass frame. Consider an expectation value of this equation within generic states, then one sees

$$\langle\psi|Q^2|\psi\rangle \sim (M + z)\langle\psi|\psi\rangle \tag{1.1.35}$$

with z an eigenvalue of Z. The left hand side is positive definite so the right hand side must be also, this implies $M \gtrsim -z$. Since the central charge matrix in full generality must be traceless (it is antisymmetric), then each positive eigenvalue has a corresponding negative eigenvalue implying

$$M \gtrsim |z| \tag{1.1.36}$$

This is the BPS bound in supersymmetry. It being saturated is equivalent to $Q|\psi\rangle = 0$; *i.e.* a state preserves some fraction, typically 1/2, of the supersymmetries. As mentioned above, one can show, though it will not be fruitful to do so here, that this definition of BPS state is totally equivalent to the definition above [67, 57]. In supersymmetric theories a saturated BPS bound then leads to zeroes in the supersymmetry algebra, this implies that the algebra has smaller than usual representations and so one obtains so-called short multiplets, more details are found in [65].

The conclusion is that BPS states in supersymmetric theories are solutions to the action which maintain some fraction of the original supersymmetry. Much of the analysis conducted later in this work will be with such states. Studying the BPS states for certain theories leads to topologically interesting solutions which have a wide range of novel properties; these are known as solitons and are the topic of the next section.

Solitons

Solitons are non-perturbative objects which arise in Yang-Mills gauge theories due to the non-trivial topological nature of the bulk spacetime. They manifest in different forms; as instantons, monopoles, vortices, and domain walls, depending on the dimension and field content of the theory. I will consider a simple soliton initially for illustrative purposes and then show how it relates to monopoles and other systems. Much of the discussion here will follow that from [68].

I work with $(4+1)$-dimensional $SU(N)$ Yang-Mills theory which can be thought of as an instanton with time dependence. The energy functional is given by

$$\mathcal{H} = \frac{1}{2g^2}\int_{\mathbb{R}^4} d^4x \mathrm{Tr} F_{MN}F^{MN}. \tag{1.1.37}$$

for $M, N = 0, 1, 2, 3, 4$. It will be useful for the moment to consider static configurations with $F_{0\mu} = 0$ for $\mu, \nu = 1, 2, 3, 4$ which implies an energy functional

$$\mathcal{H} = \frac{1}{2g^2} \int_{\mathbb{R}^4} d^4x \mathrm{Tr} F_{\mu\nu} F^{\mu\nu}. \tag{1.1.38}$$

It will turn out that the quantities of interest here are the BPS states of the system. With this in mind I run the argument of the previous section in a very similar way, this was seen in [69]. By splitting $F_{\mu\nu}$ into itself and its hodge dual where I now think in terms of the four spatial dimensions

$$\begin{aligned} \mathcal{H} &= \frac{1}{4g^2} \int_{\mathbb{R}^4} d^4x \mathrm{Tr} \left(F_{\mu\nu} \mp \star F_{\mu\nu} \right)^2 \pm 2 F_{\mu\nu} \star F^{\mu\nu} \\ &\geq \pm \frac{1}{2g^2} \int_{\mathbb{R}^4} \varepsilon^{\mu\nu\rho\sigma} \partial_\mu \left(A_\nu F_{\rho\sigma} + \frac{2i}{3} A_\nu A_\rho A_\sigma \right), \end{aligned} \tag{1.1.39}$$

where the inequality comes by dropping the positive definite term on the first line and expanding the second.

It will be interesting to see how this behaves on the boundary of space; to do this notice that a finite path integral of such a theory requires that $F_{\mu\nu}$ be zero at spatial infinity. This restricts that the gauge field, A_μ, should become pure gauge as $r \to \infty$ so

$$A_\mu \to i g^{-1} \partial_\mu g \tag{1.1.40}$$

for g an element of the Lie-algebra of $SU(N)$ so that $A_\mu \in SU(N)$. This requirement specifies a map from the boundary of space to the group $SU(N)$; such maps are specified by the homotopy of the manifold—maps are distinguished from one another by whether then can be continuously deformed into one another. For the group here, $SU(N)$, the relevant homotopy group is given by $\mathbb{Z}$. Thus integers $k \in \mathbb{Z}$, characterising how many times the map wraps over spatial infinity, will split the instanton solutions into different sectors which are totally topologically distinct. For a given map, this integer can be calculated from the Chern Class

$$k = \frac{1}{24\pi^2} \int_{S^3_\infty} d\Omega_\mu \varepsilon^{\mu\nu\rho\sigma} g^{-1} \partial_\nu g g^{-1} \partial_\rho g g^{-1} \partial_\sigma g \tag{1.1.41}$$

with Ω denoting the remaining coordinates at spatial infinity. For a discussion on Chern Classes and homotopy maps see *e.g.* [70]. Notice that the bound of equation

(1.1.39) becomes, on the boundary of space

$$\mathcal{H} \geq \frac{8\pi^2}{g^2}|k|. \tag{1.1.42}$$

The bound is then satisfied, in the same sense as before, when the manifestly positive definite terms ignored before are zero, namely when

$$F = \pm \star F \tag{1.1.43}$$

with plus or minus determined by k. This equation is the instanton equation [69]; one thus sees that instantons are states in the theory which arise from (anti)-self-dual gauge field strengths. Since, by construction, solutions to this equation minimise the action in a certain topological sector it is clear that instantons satisfy the equations of motion of the action for free. Indeed this is easy to see because

$$D_\mu(\star F)^{\mu\nu} = D_\mu F^{\mu\nu} = 0 \tag{1.1.44}$$

where the first equation is the equation of motion from the action, the first equality follows from the instanton equation, and the final equality follows from the Bianchi identity.

Finding solutions to instanton equations is an interesting process. It turns out that solutions to these equations depend on a number of extra parameters which distinguish disparate solutions. It is common to consider these parameters to be new coordinates describing a space of solutions. Such a space is in fact a manifold known as the Moduli Space, $\mathcal{M}$, with the parameters known as collective coordinates or moduli. Typically for $SU(N)$ gauge theories in the k'th sector of the theory the dimension of such a space is given by [68]

$$|\mathcal{M}| = 4kN. \tag{1.1.45}$$

How should one think about this dimension? The dimension corresponds to the number of collective coordinates of the solution. A brief example is given by $k = 1$ $SU(2)$ solutions where the collective coordinates can be intuited. Imagine the soliton as being a physical object in spacetime. Then there are eight parameters which describe a solution; four parameters for the translation invariance of its location, one parameter describing its scale size (how 'big' it is), and three parameters corresponding to the three generators of $SU(2)$ and determining its embedding into the group. Thus such an instanton has eight parameters as the equation above suggests.

The moduli space, as a manifold, has a metric which can be inherited from the field theory. The idea is that given a solution to the instanton equations, one should seek the moduli coordinates to find other nearby solutions. To do this first recognise that that any nearby solution will be accessible by a perturbation, called a zero mode, which itself must satisfy the linearised self-duality equation, $A \to A + \delta A$

$$D_\mu \delta A_\nu - D_\nu \delta A_\mu = \varepsilon_{\mu\nu\rho\sigma} D^\rho \delta A^\sigma \tag{1.1.46}$$

Zero modes are really solutions to the instanton equations which are now allowed to depend on the moduli space parameters $A_\mu = A_\mu(x^\mu, m^\alpha)$ for m^α the coordinates on the moduli space. They are defined as

$$\delta_\alpha A_\mu = \frac{\partial A_\mu}{\partial m^\alpha} + D_\mu \omega_\alpha \tag{1.1.47}$$

with the second term allowing for gauge invariance of the mode. This gauge is picked by demanding that the zero mode is orthogonal to all other gauge transformations

$$\int_{\mathbb{R}^4} \mathrm{Tr} D_\mu \Lambda \delta_\alpha A_\mu = 0 \qquad \forall \Lambda. \tag{1.1.48}$$

This implies that the gauge fixed solution satisfies $D_\mu \left(\delta_\alpha A_\mu\right) = 0$ after integrating by parts. Once such a gauge is fixed the metric can be seen to readily be the induced metric on the moduli space from the construction above

$$g_{\alpha\beta} = \frac{1}{2g^2} \int d^4x \mathrm{Tr} \left(\delta_\alpha A_\mu\right) \left(\delta_\beta A_\mu\right). \tag{1.1.49}$$

Moduli space metrics, for the examples which will be pertinent to this thesis, have various properties which also prove useful; they are Hyper-Kahler spaces with reduced holonomy which naturally inherit certain symmetries from the field theory. They are extremely valuable in assessing solitonic solutions and provides a complete description of the solitonic information in a geometric way which is often easier to compute than explicit solutions [68].

Additionally, one can now go back to the original $(4+1)$-dimensional description of the theory and consider how some time dependence on the solitons will affect this moduli space. Consider motion on the moduli space as where one takes the moduli to now have time dependence $m^\alpha = m^\alpha(t)$ which generate the motion on the space. This procedure was first described by Manton in [71] and can be though of as letting fields have the following dependence $\phi = \phi(m^\alpha(t))$ so that this should provide a good approximation for slowly moving solitons. The motion is well described by computing geodesics on the following action

$$S = \frac{1}{2}\int dt\, g_{\alpha\beta}\dot{m}^\alpha \dot{m}^\beta. \tag{1.1.50}$$

This approximation will be seen again and used explicitly in Chapter 3.

Thus far I have focused on instantons in the discussion of solitons however there are further systems which follow similar properties. By adding extra fields to the system more complex phenomena can be deduced, an example is the Georgi-Glashow model described in the previous section which described magnetic monopoles and has similarly interesting phenomena associated to it with an equivalent topological nature and moduli space [68]. One way to see this relationship is to notice that the different solitonic objects can be seen by dimensionally reducing the instanton. The argument is as follows.

Begin with the instanton equation in four dimensions and split the indices $\mu, \nu = 0, 1, 2, 3 \to i, j = 0, 1, 2$ and 3 alone. Then A splits to

$$A_\mu \to A_i, \quad X^3 \equiv A_3 \tag{1.1.51}$$

for a scalar X^4. This leads to the equation of motion, from the instanton equation

$$\begin{aligned} F_{ij} &= \varepsilon_{ijk} F^{k4} \\ &= \varepsilon_{ijk} D_k X^4. \end{aligned} \tag{1.1.52}$$

This is precisely the Bogomol'nyi equation given above in (1.1.33); thus one sees that dimensionally reduced instantons lead to monopoles.

Further, one can reduce again; perform a split $i, j = 0, 1, 2 \to \alpha, \beta = 0, 1$ and 2 and 3 distinct, so that A_i splits to

$$A_i \to A_\alpha, \quad X^2 \equiv A_2, \quad X^3 \equiv A_3. \tag{1.1.53}$$

Then the Bogomol'nyi equation becomes

$$F_{01} = [X^2, X^3], \quad D_0 X^2 = -D_1 X^3, \quad D_0 X^3 = D_1 X^2. \tag{1.1.54}$$

Which is a system of equations known as a Hitchins [72] system, the latter two equations really just enforce holomorphic conditions on a complexified system where $Z = X^2 + iX^3$ and $z = x^0 + ix^1$. Such a construction will be seen clearly in Chapter 3.

Finally, one can reduce for a final time, in precisely the same manner to obtain

$$D_0 X^1 = [X^2, X^3], \quad D_0 X^2 = -[X^1, X^3], \quad D_0 X^3 = [X^1, X^2]. \tag{1.1.55}$$

These are known as Nahm equations [73, 74] and are well known as appearing in descriptions of monopoles and other solitoninc objects.

All the solitons studied here have strong links to D-branes and this is well fleshed out in the literature; see [75] for a review. In particular there exists a construction, known as the ADHM construction [76] which highlights an explicit link between D-branes and solitons. The ADHM construction however is of primary importance because it allows for a systematic way to algebraically construct solutions to the instanton equations. The construction, from the string theory perspective [77, 78, 79], arises as one considers the bound state of Dp-branes inside D$(p+4)$-branes. Indeed instantons living in D$(p+4)$-branes are found to be precisely equivalent to Dp-branes. A completely analogous analysis also holds for magnetic monopoles and is known as the Nahm construction [80, 81]. It will not serve to relate this entire procedure here but a review is given in [68].

Gauge Theories and Systems of D-Branes

To summarise after this diversion, one should think of type II string theory as having two parts, the weakly coupled vacuum bulk theory is described by the interaction of closed string modes while the soliton like D-branes are described by fluctuations of open strings whose ends are fixed to the branes.

Going further, one can associate Lie-algebra factors to the end points of stretched open strings, in the fundamental on one end and the anti-fundamental on the other to preserve orientation. These Chan-Paton factors [82, 83, 84] allow the string to generate non-abelian gauge symmetries on D-brane worldvolumes with stacks of parallel branes. When a collection of N parallel D-branes are all non-coincident the string acts as a source for a $U(1)$ gauge symmetry on the branes on which it ends [16]. If two become coincident then the $U(1)$ factors enhance to a $U(2)$ symmetry on the brane worldvolume. In the limit as one has N coincident D-branes then one finds a $U(N)$ gauge theory living on the worldvolume of D-branes [85], in addition the representation of this gauge theory is shifted to the adjoint for coincident branes. Thus supersymmetric $(p+1)$-dimensional Yang-Mills gauge theories are typically the low energy worldvolume field theory for Dp-branes [15].

On the brane worldvolume itself one expects to find a collection of $9-p$ scalar fields to account for the transverse directions in spacetime. In the static gauge which will be generally used throughout, where one rotates the brane to coincide with $p+1$ of the bulk spacetime coordinates, then these will manifest simply as a collection X^I for $I = p+1, \ldots 10$ showing explicitly the remaining $SO(9-p)$ R-symmetry from the worldvolume perspective.

D-branes have been much studied since their discovery in string theory and many systems of multiple branes are routinely considered. Of use later will be the system of N D4-branes intersecting k separated D6-branes. This system can be seen clearly in the following way

$$\begin{array}{cccccccccccccc} N & D4 & : & 0 & 1 & 2 & 3 & 4 & & & & & \\ k & D6 & : & 0 & 1 & & & & 5 & 6 & 7 & 8 & 9 \end{array} \qquad (1.1.56)$$

where the notation makes clear that the branes intersect along the x^1 direction. This system is well studied and indeed it is well known that the open strings which stretch between the D4 and D6-branes manifest as chiral fermions found on the intersection [86]. More will be said on this topic in Chapter 5.

Having completed a brief survey of string theory including some pertinent features it is time to discuss the more modern aspirations for the theory. T-duality has been described above as providing hints at relations between the theories, this can be extended to allow for a potential unification within M-theory.

1.2 M Theory

T-duality has related the two type II string theories with one another [45]. It was also mentioned above that a similar duality exists between the heterotic strings [16], and that type I string theory can be obtained from type II via a particular projection of states [16]. One might hope then that some relation between the type II and heterotic theories might be found to tie all these disparate theories together. Such a hope will lead us to M-theory.

1.2.1 S-Duality

S-duality falls under a family of previously studied dualities in relating theories with coupling g to theories with coupling $1/g$. A classic example of this is given by electric-magnetic duality [87] by Montonen and Olive which builds on the analysis presented when discussing BPS states. One can readily see that Maxwell's equations in terms of differential forms are invariant under

$$F \rightarrow \star F \quad \star F \rightarrow -F \quad j_e \rightarrow -j_m \quad j_m \rightarrow j_e. \tag{1.2.1}$$

Similarly, it was noted when discussing the Georgi-Glashow model that the monopoles and gauge boson have masses and charges as seen in table 1.1.

Object	Mass	Charge
Magnetic Monopole	ag	$\pm g$
Gauge Boson	aq	$\pm q$

Table 1.1: Masses and charges for particles in the Georgi-Glashow Model

One sees a symmetry in both the above examples where states in the theory are invariant under the interchange of electric and magnetic charge. Under this transformation a dual description of the Georgi-Glashow model consists of gauge bosons coming from BPS magnetic monopoles, and electric monopoles.

In both cases the Dirac quantisation condition holds and so in a theory with weakly coupled, *i.e.* small, q, one must have a strongly coupled, *i.e.* big, g. Thus the duality relates a theory with weak coupling to one with strong coupling. In the particular formulation presented above in Section 1.1.4 the theory is ill defined because of possible renormalisation corrections to the BPS masses at high energy.

Supersymmetry averts this issue by fixing such quantities for all values of the coupling thus the context where this duality holds true is in $N = 2$ supersymmetric Yang-Mills theories for which a broad literature exists, particularly around so called Seiberg-Witten theory [88, 89] with richer symmetry transformations than the simple $\mathbb{Z}_2$ presented here.

Such weak-strong dualities arise in string theory as well and are referred to generally as S-dualities. It will not serve to fill in all the details here but as a flavour, the heterotic string is shown dual to the type I in [90, 91]. Additionally the type IIB theory is shown to be self dual in an S-dual sense in [92]. A complex web of such dualities has arisen over the years tying the various string theories together and into the subject of the next section—M-theory. These dualities are conjectures, but they are conjectures for which there is good reason to believe that they are true dualities. Reviews of this web can be found in [93, 94].

With this in place, one might start to seriously wonder at the extent to which we should think of the supposedly disparate string theories as being the same. M-theory now takes centre stage as a unifying theory to encapsulate these string theories; one key insight demonstrating this comes from the relationships between supergravities in ten and eleven dimensions.

1.2.2 Supergravity and M-Theory

Thus far in describing type IIA and IIB string theory the states considered have been the lowest energy excitations of the theory. This is a good approximation for the theory at weak coupling where we consider l_s small or equivalently large string tension. This weak coupling field content is also known as type IIA supergravity; a well studied theory in its own right first obtained by the dimensional reduction related below [95]. An equivalent statement can be made about type IIB string theory having low energy field content of type IIB supergravity [96, 97, 98]. Together these constitute the maximally supersymmetric supergravity theories in ten dimensions.

In eleven dimensions there is only one possible supergravity theory with maximal supersymmetry [99]; indeed eleven dimensions is the maximal number of dimensions in which a supergravity is realised without any fields of spin greater than 2 [100]. This is typically believed to act as a cut off for realistic theories as those with higher spin fields have inherent pathologies, though there is ongoing study to see if there is any remedy for this. The field content for eleven-dimensional supergravity is very simple, it contains a graviton, G_{MN}, a corresponding spin 3/2 fermionic gravitino, Ψ_M, and, as can be deduced from matching degrees of freedom, a three-form field, A_3 with corresponding field strength $F_4 = dA_3$. The bosonic action takes the form [99]

$$S = \frac{1}{16\pi G_{11}} \left[\int \sqrt{-G} \left(R^{(11)} - \frac{1}{2 \cdot 4!} F^{MNPQ} F_{MNPQ} \right) - \frac{1}{6} \int A_3 \wedge F_4 \wedge F_4 \right], \tag{1.2.2}$$

with G_{11} the eleven-dimensional gravitational constant and $R^{(11)}$ the Ricci scalar as before. Throughout, conventions are used where $M, N, \ldots$ represent eleven-dimensional indices while $\mu, \nu \ldots$ represent ten-dimensional indices. The action is invariant under supersymmetry, general coordinate transformations, and a three-form gauge symmetry $\delta A_3 = d\Lambda_2$. A key feature of this model is the implication, following previous analysis, that the theory contains extended p-brane like objects due to the presence of the three-form [101, 102]. In this case the three-form generates electrically an object known as an M2-brane and magnetically an M5-brane, these both saturate BPS bounds on their tensions and satisfy the equations of motion for the theory. Thus they are stable brane solutions to eleven-dimensional supergravity which, generically, preserve half the supersymmetry. Their tensions can be calculated by imposing suitable symmetry and dimensional constraints, the results are found to depend only on the eleven-dimensional Planck length, the only parameter in the theory, and are given by [55]

$$T_{M2} = \frac{2\pi}{(2\pi l_p)^3} \qquad T_{M5} = \frac{2\pi}{(2\pi l_p)^6}. \tag{1.2.3}$$

The key idea which will extend our thinking from string theory to M-theory is in the relation between this theory and its ten-dimensional reduction. One follows the standard Kaluza-Klein prescription laid out earlier where we take $x^{10} \sim x^{10} + 2\pi R$, so that the eleven-dimensional graviton splits to a ten-dimensional graviton, a one-form gauge field, and a dilaton. In addition we have an eleven-dimensional three-form field which in this case splits to

$$C^{(10)}_{\mu\nu\rho} = A^{(11)}_{\mu\nu\rho} \qquad B_{\mu\nu} = A^{(11)}_{\mu\nu 10}. \tag{1.2.4}$$

One also sees that the gauge symmetry acting on the three-form in eleven dimensions translates to impose a gauge symmetry on the induced forms in ten dimensions. This means that the full bosonic field content of the dimensionally reduced theory is given by a graviton, a two-form gauge field, a dilaton, a one-form gauge field, and a three-form gauge field. This is precisely the massless bosonic spectrum of type IIA string theory, or equivalently of ten-dimensional type IIA supergravity. The supersymmetry in eleven-dimensional supergravity consists of a single 32 component Majorana spinor. Upon reduction this can be decomposed into two Weyl spinors of opposite chirality, this is precisely the fermionic content described for type IIA string theory earlier. The reduction of the graviton allows one to relate the Planck length in eleven dimensions to the string coupling constant, dynamically generated by the dilaton, and the string length scale. The radius of compactification is also set by these quantities so that in total [16]

$$l_p = l_s g_s^{1/3} \qquad R = l_s g_s. \tag{1.2.5}$$

From this, some conclusions can be drawn. Eleven dimensional supergravity admits a dimensional reduction to type IIA supergravity. Type IIA supergravity is itself the low energy effective action of type IIA string theory, obtained by considering the theory at small g_s, or equivalently just taking the lowest level, massless excitations of the string spectrum. Thus it's clear from the ten-dimensional perspective that the supergravity theory has a full UV completion given by type IIA string theory. M-theory can be phrased as the proposition that there exists a well defined strong coupling limit of eleven-dimensional supergravity whose dimensional reduction is type IIA string theory, and to which we attach the mysterious name of M-theory. This argument, that type IIA string theory becomes eleven-dimensional at strong coupling was originally made by Townsend and Witten in [103, 90, 104]

One might be concerned at this point about the brane content of M-theory. The branes are all BPS objects and so one does not expect them to change in character at strong coupling, yet we find only the M2 and M5-brane in M-theory where type IIA string theory has a wide selection of even Dp-branes. The resolution here is to consider how the compactification of the M-theory circle interacts with the branes [103, 90, 104, 105]. If the circle is aligned along either brane, this direction on the brane becomes very small, and ultimately disappears in the full string theory limit; thus the M-branes become an F1 fundamental string and a D4-brane respectively. Similarly, if the M-branes are perpendicular to the circular compactification then they are unaffected by the reduction and so reduce simply to a D2 and the NS5 brane. By performing a simple calculation one can see that the tension of the M2 and M5-brane in these cases reduce precisely to the corresponding D-brane and F1/NS5 tensions given earlier [55]. One might worry that the F1 and NS5 branes cannot have their tensions analysed in this way as they are potentially changed in the strong coupling M-theory regime. A result in [106] ensures that, due to them being maximally supersymmetric, their tensions are not changed at strong coupling so that the M-theory matching is legitimate. This is a key non-perturbative result suggesting the M-theory conjecture is correct. Thus one sees that the M-theory branes can reproduce the fundamental string, NS5-brane, D2-brane, and D4-brane simply. This leaves the D0 and D6-branes to explain.

In any Kaluza-Klein compactification there are the KK modes arising from momentum running around the circle. These states have mass given by $M = \frac{N}{R}$ as described earlier. In the M-theory compactification R is determined by the string length and coupling (1.2.5) so that the first excited KK mode has mass, or tension,

$$T = \frac{1}{l_s g_s} \tag{1.2.6}$$

this is precisely the tension of the D0-brane, implying that D0-branes arise simply from the KK momentum running around the M-theory circle.

The D6-brane arises as the magnetic dual to the D0-brane in string theory and is thus magnetically charged under the $U(1)$ one-form gauge field which electrically couples to the D0-brane. Thus one seeks to understand the geometry which gives rise to this magnetic coupling. This is a well known problem whose solution amounts to finding a metric describing a Kaluza-Klein magnetic monopole [107, 108]. The geometry which produces such a monopole is known as a Taub-NUT space [109], it is constructed as a circle fibration over a warped 3 dimensional space with metric

$$ds^2_{TN} = H(\vec{x}) d\vec{x} \cdot d\vec{x} + \frac{1}{H(\vec{x})} \left(dy + \vec{A} \cdot d\vec{x} \right)^2 . \tag{1.2.7}$$

$\vec{x}$ defines a 3 component vector, A is a vector potential for a magnetic monopole, $\vec{B} = \nabla \times \vec{A}$, y is a periodic coordinate, and H is a harmonic function constrained by $\vec{B} = -\nabla H$.

Given the choice of harmonic function

$$H(\vec{x}) = 1 + \frac{R}{2r} \tag{1.2.8}$$

for r the natural radius on the $\mathbb{R}^3$, this leads to a magnetic field

$$|\vec{B}| \sim \frac{1}{r^2} \tag{1.2.9}$$

as one would expect for a magnetic monopole solution. The Taub-NUT metric is everywhere smooth, even near the soliton core at $r \to 0$, this is ensured by the periodicity of the y coordinate. One can perform a simple calculation to compute the energy density of this space to find that it agrees precisely with the tension for a D6-brane [55]. Thus the interpretation is that one takes this space and embeds it into M-theory so that the total geometry is given by $TN \times \mathbb{R}^{1,6}$ and this soliton geometry then acts as the source for string theory D6-branes [15].

The Taub-NUT geometry is interesting in its own right as a solution to eleven-dimensional supergravity and for its properties as describing magnetic monopoles. It can also be extended to a multi Taub-NUT space describing a collection of N monopoles with various centres, which will be seen in Chapter 5.

It has now been shown that there exists a correspondence between the objects of M-theory and those of type IIA string theory. Furthermore I have intimated a collection of dualities which link the various string theories together. From this point it is natural to realise that from any string theory one can, by some chain of dualities, arrive at the type IIA theory and lift to M-theory.

Furthermore one doesn't even need to dualise to type IIA to reach the M-theory limit. For example, from T-duality, one can consider the type IIB theory to be simply M-theory defined on a torus. In addition there are well studied distinct lifts to M-theory which can be made directly from the heterotic theories without any reference to type IIA; one simply finds that in a particular limit the heterotic theory exhibits an extra dimension with appropriate properties to be identified as the same M-theory [110, 84]. Thus, the proposal is that the string theories are all manifestations of the same underlying eleven-dimensional theory which is known as M-theory.

The proposed relationship between M-theory and string theory is now well established and studied. One of the key challenges in the contemporary study of M-theory is in describing how the two objects which live in the theory, the M2-brane and M5-brane, interact. In particular one would like to understand better how to build models of multiple branes, this subject has matured for the M2-brane but is largely unknown for the M5-brane; this will be the subject of the next chapter.

Chapter 2

Non-Abelian Theories of M2 and M5-branes

In this chapter I will review some modern developments of M-branes in M-theory. Developing a greater understanding of how these objects arise and interact in the full theory is a topic of intense study and this thesis aims to add to the discussion in some part.

As described above, M-theory is a proposed realisation of the unification of the five distinct string theories into one overarching theory. At weak coupling it reduces to the well studied eleven-dimensional supergravity but at strong coupling its formulation is highly mysterious, even after years of work. In addition we can infer the existence of two key objects in the theory: the M2-brane and its magnetic dual, the M5-brane. Like the D-branes of string theory, these branes can be well studied by looking at the theories which are found on their worldvolume. This process is relatively straightforward for M2-branes as the fields which are present are well understood and there are correspondingly well developed theories to describe these worldvolumes. In the M5-brane case things are more complicated because the presence of a self-dual field strength on the worldvolume means that an action cannot be simply written down.

This chapter will be structured in the following way; in the first section I will provide an overview of our modern understanding of M2-branes, starting with the BLG theory describing a pair of M2-branes which I discuss in depth. I will then, for completeness, briefly mention the ABJM theory which acts as an extension to

BLG for systems of N M2-branes. The ABJM model however is not necessary for the work in later chapters so I do not dwell long on its construction. In the second section I provide an overview of the current state of work in understanding M5-branes. The worldvolume of multiple M5-branes is believed to be described by the six-dimensional $(2,0)$ theory, however this is still poorly understood apart from in the free abelian case describing a single M5-brane. In the final section I consider extensions of the abelian six-dimensional $(2,0)$ multiplet where non-abelian gauge symmetry is introduced. This process is a relatively recent area of study and can lead to interesting configurations of M2 and M5-branes.

2.1 M2-branes

M2-branes are one of two extended solitonic objects found in the full eleven-dimensional theory. They can be seen as giving rise to either the fundamental string or D2-branes from the ten-dimensional string theory perspective, depending on their orientation with respect to the M-theory circle. Writing down an action for a single M2-brane is a relatively straightforward procedure. Setting the three-form of eleven-dimensional supergravity to zero for simplicity, and taking the static case where the first three spacetime brane coordinates, X^μ for $\mu = 0,1,2$, coincide with the worldvolume coordinates along the brane x^μ, one finds the simple action for a single M2-brane as [111]

$$S = -T_{M2} \int d^3x \sqrt{-\det\left(\eta_{\mu\nu} + \frac{1}{T_{M2}} \partial_\mu X^I \partial_\nu X^I\right)}. \tag{2.1.1}$$

The convention is that $I, J, = 3, \ldots 10$ represent the transverse coordinates off the brane.

This construction is very simple and has been understood for a long time. Moving forward to describe multiple branes proved trickier, one reason is that a system of N M2-branes, being the strong coupling limit of a system of N D2-branes, should be expected to be some fixed point of a $U(N)$ maximally supersymmetric Yang-Mills theory. For the case of relating a single M2-brane to a single D2-brane one can consider the appropriate DBI action [52] for the D2-brane and take the coupling $g_{YM} \to \infty$; upon doing this the M2-action above is recovered exactly as expected and thus justifying (2.1.1). A similar process for a system of N D2-branes implies a non-abelian Yang-Mills theory whose strong coupling limit is harder to discern. This is precisely the action sought in this section and once obtained, it will be justified by arguing that it reduces to the appropriate non-abelian theory for D2-branes.

Progress was made in seeking such an M2-action by studying systems of brane intersections in string theory and using dualities to lift to their M-theory counterparts. An example of such early progress is given by the work of Basu and Harvey [112]. They began with the well understood system of a D1 brane intersecting a D3 brane in type IIB string theory. This system can be roughly viewed as a D3-brane with a fuzzy soliton like spike intersecting it, or equivalently a D1-brane with a fuzzy sphere in the theory. The system is described by the solitonic Nahm equations. Furthermore, this system is intimately related to a similar setup in M-theory; if one T-dualises off the branes, one obtains the D2-D4 system in IIA, then lifting to M-theory in a direction along the D4 but not the D2, one finds a system with an M2-brane intersecting an M5-brane [113]. This fact allows one to intuit some features necessary in the construction of a generalised theory and indeed the equations governing this system are found to resemble those in the initial system where we now think of the M5-brane has having a kind of fuzzy solitonic ridge on its worldvolume corresponding to the M2-brane. This generalisation also necessitated a new kind of triple product, an addition that would find itself entrenched into the description of M2-branes with the BLG theory.

2.1.1 BLG Theory

The BLG theory was the first theory to provide a consistent description for a pair of M2-branes. It was developed in [114, 115, 116, 117] by Bagger, Lambert, and Gustavsson. I will review the construction in detail as it forms the basis for modern thinking about M2-branes and many features developed here will be seen later in this work. Much of the analysis presented here will follow that in [55] which provides a comprehensive review of systems of multiple M2-branes.

The M2-brane is a solitonic like BPS object and as such preserves half of the 32 spacetime supersymmetries. This implies that one should look for $N = 8$, $(2 + 1)$-dimensional worldvolume theories. It is also clear *a priori* that M2-branes will break the spacetime Lorentz symmetry $SO(1, 10) \rightarrow SO(1, 2) \times SO(8)$. The $SO(8)$ will be visible as a collection of 8 scalar fields corresponding to the spacetime embedding of the brane.

The supersymmetry is characterised by a parameter ϵ which I take as a 32 component Majorana spinor, the BPS condition on the brane enforces the following condition

$$\Gamma_{012}\epsilon = \epsilon. \tag{2.1.2}$$

I work with full eleven-dimensional Majorana spinors rather than the usual 2 component spinors as would typically be done for $(2+1)$-dimensional theories.

The simplest possible choice of multiplet for the theory thus contains 8 scalar modes as described above, denoted X^I, $I = 3, \dots, 10$ and a fermion Ψ which has 32 components, halved to 16 on-shell by the Dirac equation, and which are really Goldstino modes of the supersymmetry breaking and thus satisfy

$$\Gamma_{012}\Psi = -\Psi. \tag{2.1.3}$$

This reduces further the number of on-shell components to 8. This matching of fermioninc and bosonic degrees of freedom implies a full supersymmetry multiplet. It does not however exclude the possibility of non-dynamical modes in the theory, in particular gauge modes.

I will derive out the BLG model from this basis. Place the fields in some algebra with generators

$$\begin{aligned} X^I &= X^I_A T^A \\ \Psi &= \Psi_A T^A \end{aligned} \tag{2.1.4}$$

A natural choice will turn out to be that $A = 1, 2, 3, 4$. Then one can guess supersymmetry conditions acting on the fields which will generate some dynamical behaviour

$$\begin{aligned} \delta X^I_D &= i\bar{\epsilon}\Gamma^I \Psi_D \\ \delta \Psi_D &= \partial_\mu X^I_D \Gamma^\mu \Gamma^I \epsilon - \frac{1}{3!} f^{ABC}{}_D X^I_A X^J_B X^K_C \Gamma^{IJK}\epsilon. \end{aligned} \tag{2.1.5}$$

The first term in each transformation is the standard supersymmetry transformation. Greek characters $\mu, \nu = 0, 1, 2$ represent coordinates along the brane since I work in static gauge. The second term in the fermion variation is introduced to induce interactions in the simplest possible way. The variations must satisfy the chirality conditions (2.1.2), and (2.1.3); thus, since the conditions have opposite sign, such a cubic term is the simplest possible choice. The $f^{ABC}{}_D$ are structure constants of the algebra which are totally anti-symmetric in the first three indices. I pause here to briefly discuss this new structure.

It is natural to consider this relation as a generalisation of a typical Lie-algebra so that one defines a triple product to be

$$[T^A, T^B, T^C] = f^{ABC}{}_D T^D. \tag{2.1.6}$$

This new structure is referred to as a Lie 3-algebra, or 3-algebra for short. Such structures have been studied in the mathematical literature extensively but this is one of the first times the structure has been employed in physics. Recent discussions of their properties are given in numerous publications, see [118, 119, 120]. Like a Lie-algebra, one can think of the Lie 3-algebra as a vector space, $\mathcal{V}$, with canonical product

$$[\cdot,\cdot,\cdot] : \mathcal{V}\times\mathcal{V}\times\mathcal{V}\to\mathcal{V} \tag{2.1.7}$$

which is linear in all entries and satisfies an extension of the Jacobi identity

$$[U, V, [X, Y, Z]] = [[U, V, X], Y, Z] + [X, [U, V, Y], Z] + [X, Y, [U, V, Z]] \ , \tag{2.1.8}$$

called the fundamental identity which ensures that the algebra is closed. In terms of the structure constants, this condition reads

$$f^{[ABC}{}_E f^{D]EF}{}_G = 0. \tag{2.1.9}$$

There also exists an inner product on this space, needed to construct actions, given by

$$\langle X, Y\rangle = h^{AB} X_A Y_B \tag{2.1.10}$$

with h acting as a metric on the structure so that

$$f^{ABCD} = h^{DE} f^{ABC}{}_E. \tag{2.1.11}$$

In this form the structure constant is totally antisymmetric in all indices, $f^{ABCD} = f^{[ABCD]}$. Throughout this thesis the inner product will be defined by $h^{AB} = \delta^{AB}$.

Additionally, there exists a 3-algebra analogue of the adjoint action which generates a Lie-algebra, $\mathcal{G}$, as $\varphi_{UV}(X) = [U, V, X]$ for all $U, V \in \mathcal{V}$. On this Lie-algebra then one finds another inner product $(\cdot, \cdot)$ which acts as

$$(T, \varphi_{UV}) = \langle T(U), V \rangle \tag{2.1.12}$$

for $T \in \mathcal{G}$ and $U, V \in \mathcal{V}$. This formulation will be seen explicitly in Chapter 4.

With such a structure in place one can check for the closure of the supersymmetry algebra (2.1.5). On the scalars one finds the usual translation term with an extra, coordinate dependent term. Since this term has coordinate dependence, it must represent an additional gauge symmetry. A symmetry acting on a parameter in the 3-algebra must be a two index object as $\delta X = [A, B, X]$. Thus one introduces a gauge field, $\tilde{A}_\mu{}^A{}_B$, to generate this symmetry in a covariant derivative

$$D_\mu X^I_B = \partial_\mu X^I_B - \tilde{A}_\mu{}^A{}_B X^I_A. \tag{2.1.13}$$

Notice that in the language of the above, this gauge field is an element of the Lie-algebra, $\mathcal{G}$, and so one could rephrase (2.1.12) very explicitly as

$$(A, [U, V, \cdot]) = \langle A(U), V \rangle . \tag{2.1.14}$$

The derivative is gauge covariant if

$$\delta \tilde{A}_\mu{}^A{}_B = D_\mu \Lambda^A{}_B \tag{2.1.15}$$

with $\Lambda^A{}_B$ an infinitesimal gauge symmetry parameter. That this parameter is antisymmetric in its indices, and that there are four indices, implies that the gauge symmetry found here is $SO(4) \simeq SU(2) \times SU(2)$.

The field strength tensor associated to this can be constructed by taking

$$\tilde{F}_{\mu\nu} = -[D_\mu, D_\nu] \tag{2.1.16}$$

as usual. I note here that the gauge structure described above is precisely the same as found for an ordinary gauge theory under a Lie-algebra with the gauge field in the adjoint represetation.

The supersymmetry transformation of the gauge field can be guessed based on canonical dimensions and index structure to give the following set of final supersymmetry transformations

$$\begin{aligned}
\delta X^I_D &= i\bar{\epsilon}\Gamma^I\Psi_D \\
\delta\Psi_D &= \partial_\mu X^I_D\Gamma^\mu\Gamma^I\epsilon - \frac{1}{3!}f^{ABC}{}_D X^I_A X^J_B X^K_C \Gamma^{IJK}\epsilon \\
\delta\tilde{A}_\mu{}^A{}_B &= i\bar{\epsilon} f^{CDA}{}_B X^I_C \Gamma_\mu \Gamma_I \Psi_D.
\end{aligned} \tag{2.1.17}$$

One finds that demanding these transformations close completely is extremely prescriptive and in fact totally specifies the equations of motion for the system

$$\begin{aligned}
0 &= D_\mu D^\mu X^I - \frac{i}{2}[\bar{\Psi}, X^J, \Gamma^{IJ}\Psi] + \frac{1}{2}[X^J, X^K, [X^I, X^J, X^K]] \\
0 &= \tilde{F}_{\mu\nu}(\cdot) + \varepsilon_{\mu\nu\rho}\left([X^I, D^\rho X^I, \cdot] + \frac{i}{2}[\bar{\Psi}, \Gamma^\rho\Psi, \cdot]\right) \\
0 &= \Gamma^\mu D_\mu \Psi + \frac{1}{2}[X^I, X^J, \Gamma_{IJ}\Psi].
\end{aligned} \tag{2.1.18}$$

The dot in the gauge field equation of motion is simply indicating that the field strength naturally acts on 3-algebra valued fields and so can be defined on any such field. In the limit where the interactions are zero, *i.e.* the 3-brackets are set to zero, these equations reduce to the massless Klein-Gordon equation, the free Dirac equation, and a flat connection. That we obtain a flat connection rather than the equation of motion for a Yang-Mills field is a consequence of the fact that this gauge field is non-dynamical. This is good news since the scalars and fermions constitute a full supersymmetry multiplet on their own.

Next I wish to write down an appropriate action for this system. Since the gauge field reduces to a flat connection, one should not expect a typical Yang-Mills term for the gauge field but, rather, the gauge field should enter through a Chern-Simons-like term.

Chern-Simons theories [121] are (2+1)-dimensional topological quantum field theories so called because their actions are given by integrals over the three-dimensional Chern-Simons form which is an important object in the theory of characteristic classes of gauge fields [70]. They are defined for a gauge field A by the action

$$S = \frac{k}{4\pi}\int A \wedge dA + \frac{2}{3}A \wedge A \wedge A. \tag{2.1.19}$$

The level integer k is quantised [122] since the action is not invariant under large gauge transformations—those which are homotopic to the identity. This is seen because under such gauge transformations the action changes, at least for $SU(N)$ gauge groups, as $S \to S + 2\pi k w$ where $w \in \mathbb{Z}$ is known as the winding number of the gauge transformation. Roughly, it measures how many times the gauge transformation wraps the space. Since in the full quantum theory one really cares about the path integral, requiring that $\exp(iS)$ is invariant fixes the level $k \in \mathbb{Z}$. Chern-Simons theories can be coupled to matter and generally imply chiral sectors of theories because of the parity breaking cubic term. In particular, pure Chern-Simons theory, when defined on a manifold with boundary [123], gives rise to the so called Wess-Zumino-Witten (WZW) theory [124] which will be an important part of the analysis in Chapter 5.

With this in mind I now write down the full BLG lagrangian [117]

$$\mathcal{L} = -\frac{1}{2}\left\langle D_\mu X^I, D^\mu X^I\right\rangle + \frac{i}{2}\left\langle \bar{\Psi}, \Gamma^\mu D_\mu \Psi\right\rangle + \frac{i}{4}\langle \bar{\Psi}, \Gamma_{IJ}[X^I, X^J, \Psi]\rangle - V + \mathcal{L}_{CS} \quad (2.1.20)$$

with

$$V = \frac{1}{2\cdot 3!}\left\langle [X^I, X^J, X^K], [X^I, X^J, X^K]\right\rangle \quad (2.1.21)$$

and

$$\mathcal{L}_{CS} = \frac{1}{2}\varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(f^{ABCD}A_{\mu\,AB}\partial_\nu A_{\rho\,CD} + \frac{2}{3}f^{CDA}{}_G f^{EFGB}A_{\mu\,AB}A_{\nu\,CD}A_{\rho\,EF}\right) \quad (2.1.22)$$

where $\tilde{A}_\mu{}^A{}_B = A_{\mu\,CD}f^{CDA}{}_B$ has been introduced for convenience in the "twisted" Chern-Simons lagrangian. One can check that this lagrangian is supersymmetric under the transformations (2.1.17) up to a total derivative. One can also see plainly that the required spacetime symmetries, $SO(1,2)\times SO(8)$ are satisfied, and by suitable definitions on the parity transformations of the fields, the lagrangian can be made totally parity invariant.

Note that the whole lagrangian has no free parameters aside from the normalisation of the structure constants. One should therefore expect that different choices of solutions for the structure constants should modulate the theory between different M2 configurations and in particular we should expect these choices to be quantised by the Chern-Simons argument above.

However, the amount of supersymmetry in the system is highly constraining and indeed the requirement that the inner product on the 3-algebra be positive definite—

thus that energies are positive definite—enforces a fixed solution on the structure constants [125, 126, 127], namely

$$f^{ABCD} = \frac{2\pi}{k}\varepsilon^{ABCD} \quad \text{with} \quad h^{AB} = \delta^{AB}. \tag{2.1.23}$$

This fix means that the BLG system does not describe arbitrary systems of M2-branes, instead it must describe a subset. To summarise, I have reviewed the existence of a supersymmetric $SU(2) \times SU(2)$, $(2+1)$-dimensional theory which has all the expected symmetry properties for M2-branes and is totally fixed up to a level integer k.

This split of the gauge field as $SO(4) \simeq SU(2) \times SU(2)$ is made manifest by splitting the gauge field into self and anti-self dual components; one can then introduce pauli matrices for each $SU(2)$ piece to reduce the Chern-Simons lagrangian to

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}\left[\left(A^L_\mu\partial_\nu A^L_\rho - \frac{2i}{3}A^L_\mu A^L_\nu A^L_\rho\right) - \left(A^R_\mu\partial_\nu A^R_\rho - \frac{2i}{3}A^R_\mu A^R_\nu A^R_\rho\right)\right], \tag{2.1.24}$$

i.e. the difference of two standard Chern-Simons lagrangians. A particular choice of parity action on the structure constants, as mentioned above, enforces that under parity the lagrangian is invariant as $R \leftrightarrow L$ [128, 129].

It remains to understand the interpretation of this theory as describing a pair of M2-branes. One way to see this was first given in [130]; I will outline the procedure here. The setup is to consider the Higgs mechanism for the theory and show that this reduces the model to describing a pair of D2-branes. To begin I take one of the scalar fields to acquire a vacuum expectation value (vev), and then use the $SO(4)$ rotational symmetry to rotate this to act only in one component of the internal algebra. Thus one can take, say $\langle X^{I=8\,A=4}\rangle = v$, where this will cause a split in the $SO(4)$ index to $A = 1, 2, 3$ and 4 alone. This choice will not break any supersymmetry as one can see from (2.1.17). In all 3-algebra equations then the epsilon tensor will naturally split to $\varepsilon^{ABC4} = \varepsilon^{ABC}$ which essentially reduces terms to usual Lie-algebra terms. For example if one does this to the potential then one finds that the remaining scalars X^i, $i = 1, \ldots 7$ reduce to

$$V \to \frac{1}{2}v^2\left\langle [X^i, X^j], [X^i, X^j]\right\rangle \tag{2.1.25}$$

which is precisely the quartic interaction for maximally supersymmetric $(2+1)$-

dimensional $SU(2)$ Yang-Mills theory [55]. A similar result holds in that the fermion-scalar coupling in BLG reduces to the usual Yukawa coupling under this vev.

The initial difficulty appears to be, how to recover the Yang-Mills kinetic term from a theory with a non-dynamical gauge field entering only through a Chern-Simons term. This is resolved through the splitting of the gauge field. Under this Higgs setup the gauge field A_μ^{AB} splits to A_μ^A and B_μ^A as

$$A_\mu^{A4} \equiv A_\mu^A \quad \text{and} \quad \frac{1}{2}\varepsilon^A_{\ BC} A_\mu^{BC} \equiv B_\mu^A. \tag{2.1.26}$$

One defines a natural covariant derivative and field strength based on the A_μ field and subsequently finds that the B_μ field has no derivative terms and so acts as a constraint field only. Upon solving the equations of motion for this field one finds that

$$B_\mu \sim \varepsilon_\mu^{\ \nu\rho} F_{\nu\rho} + \ldots \tag{2.1.27}$$

with F the natural field strength associated to the new A_μ field and the ellipsis hiding extra terms which are important but not worth making explicit here. A miracle then occurs because the lagrangian contains a $B_\mu B^\mu$ term, which produces the required Yang-Mills field strength. In total, after some work and rescaling, one obtains the following lagrangian [130]

$$\mathcal{L} = \mathcal{L}_{SU(2)} + \mathcal{L}_{U(1)} \tag{2.1.28}$$

where

$$\mathcal{L}_{U(1)} = -\frac{1}{2}\partial_\mu X^{I\,4} \partial^\mu X_4^I + \frac{i}{2}\bar{\Psi}^4 \Gamma^\mu \partial_\mu \Psi_4 \tag{2.1.29}$$

gives an additional U(1) symmetry to the theory and

$$\begin{aligned}\mathcal{L}_{SU(2)} = \frac{1}{v^2}\Bigg[&-\frac{1}{4}\langle F_{\mu\nu}, F^{\mu\nu}\rangle - \frac{1}{2}\left\langle D_\mu X^i, D^\mu X^i\right\rangle + \frac{1}{4}\left\langle [X^i, X^j], [X^i, X^j]\right\rangle \\ &+ \frac{i}{2}\left\langle \bar{\Psi}, \Gamma^\mu D_\mu \Psi\right\rangle + \frac{i}{2}\left\langle \bar{\Psi}, [X^i, \Gamma^i \Psi]\right\rangle\Bigg] + \mathcal{O}\left(\frac{1}{v^3}\right).\end{aligned} \tag{2.1.30}$$

The part proportional to $1/v^2$ is precisely $(2+1)$-dimensional $SU(2)$ maximally supersymmetric Yang-Mills theory—the low energy theory describing a pair of D2-branes. As $v \to \infty$ then in the interacting theory only this part survives and v can be identified $v = g_{YM}$, with the coupling of the Yang-Mills theory. At weak coupling this theory describes a pair of D2-branes; it is expected that at strong coupling this should lift to a pair of M2-branes. It is for this reason that the BLG theory presented above is interpreted as describing a pair of M2-branes [130, 55].

Finally it is worth pointing out how this result was achieved. The Higgs mechanism broke the $SO(4)$ symmetry of the original theory to $SO(3) \simeq SU(2)$ with the scalar modes $X^{8\,A}$ left over to promote the gauge field to a dynamical gauge field. There are also a set of scalars $X^{I\,4}$ which give the seven-dimensional centre of mass position of the D2-branes, and the additional $U(1)$ field. Thus the system found here has full gauge group $SU(2) \times U(1)$.

There is a natural extension to the BLG theory presented above. It was discovered by Aharony, Bergman, Jafferis, and Maldacena in [131], and is commonly referred to as ABJM theory. The trick to extending the previous analysis lies in recalling that BLG could only describe a pair of M2-branes because the theory was highly constrained by the requirements imposed on it. So a natural guess is to relax some of the stringent requirements. The route taken to obtain the ABJM model is to reduce amount of supersymmetry in the theory. In particular, the ABJM model has only $N = 6$ supersymmetry. With this change the construction is quite similar and a $U(N) \times U(N)$ theory with level number k is found. The physical interpretation is found to be systems of N M2-branes on a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold. The ABJM theory reduces to BLG in the cases where $N = 2$ and $k = 1$; this provides another check on the argument that BLG theory describes a pair of M2-branes. A thorough review of this theory can be found in [55]. The ABJM model will not be necessary for the work discussed later in this thesis and so I will not describe it in any more depth here.

There are however more recent systems which have been found to describe M2-branes; such systems arise from the six-dimensional $(2,0)$ theory and are important for the rest of this work. They are discussed in Section 2.3. Principally however the $(2,0)$ theory is thought to describe the worldvolume theory for multiple M5-branes and it is this perspective which the next section will discuss.

2.2 M5-branes

The other BPS extended object found in M-theory is the M5-brane [101]. The worldvolume arising on such a brane is far more mysterious than for that on the M2-brane, though the basic components of the worldvolume theory can be guessed relatively simply. It should include five scalar fields (by analogy with the M2's eight) which arise as the embedding transverse coordinates of the brane. In this section and the next, lower case latin characters will be used for coordinates off the brane in anticipation for the discussion of Chapter 3 so X^i for $i = 6, 7, 8, 9, 10$. In addition, the fact that D-branes in string theory provide the end point of fundamental strings is of importance here; in particular that F1 strings end on D4-branes. This is seen in the worldvolume field theory as a one-form gauge field which carries away the associated flux from the string. The M-theory realisation of this fact is that M2-branes can end on M5-branes and in this context it is therefore natural to expect a two-form gauge field, B_2, on the M5-brane worldvolume which carries away the associated flux [16]. Thus, one would expect the bosonic content of the M5-brane worldvolume to consist of five scalar fields and a two-form gauge field. Counting degrees of freedom then implies that this two-form should have a self dual field strength, $H = \star H$ with $H = dB$, so that in total there are eight on-shell bosonic degrees of freedom to match the eight on-shell fermionic degrees of freedom (which arise from the same argument in Section 2.1.1). This self duality is also seen due to the existence of a self-dual string soliton, sourced by this field, living on the worldvolume of the theory, this was first seen in [132] and further studied in [113, 133].

One reason why M5-branes are considered far more mysterious than M2-branes is that M5-branes do not admit an obvious lagrangian description [134]. The primary cause of this difficulty is the presence of the self dual three-form on the worldvolume. In any theory with such a self dual form the kinetic term is ill defined—this is because a kinetic term for such a field must be of the form

$$S \sim \int H \wedge \star H = \int H \wedge H = 0 \tag{2.2.1}$$

where the vanishing is due to the anti-symmetry properties of the wedge product acting on three-forms: $A_3 \wedge B_3 = -B_3 \wedge A_3$. One can find various actions for theories with self dual forms, where self-duality is imposed by the introduction of an auxiliary scalar field however these lagrangians are not manifestly Lorentz invariant and a path integral quantisation of such theories is not clear [135, 136, 137, 138, 139].

In lieu of this difficulty typically one works directly with the equations of motion and supersymmetry transformations of the theory. For a single M5-brane the dynamical equations have been known for some time [140, 135, 141, 137], as a six-dimensional abelian tensor multiplet. At lowest order, in the decoupling limit, this reduces to a free field theory in flat space:

$$\begin{aligned} \partial^2 X^i &= 0 \\ i\Gamma^\mu \partial_\mu \Psi &= 0 \\ H_{\mu\nu\rho} &= \frac{1}{3!}\varepsilon^{\mu\nu\rho\sigma\lambda\tau} H_{\sigma\lambda\tau} \end{aligned} \tag{2.2.2}$$

with supersymmetry transformations

$$\begin{aligned} \delta X^i &= i\bar{\epsilon}\Gamma^i \Psi \\ \delta B_{\mu\nu} &= i\bar{\epsilon}\Gamma_{\mu\nu}\Psi \\ \delta\Psi &= \partial_\mu X^i \Gamma^\mu \Gamma^i \epsilon + \frac{1}{2\cdot 3!}\Gamma^{\mu\nu\rho} H_{\mu\nu\rho}\epsilon. \end{aligned} \tag{2.2.3}$$

In this case $\mu, \nu = 0, 1, 2, 3, 4, 5$ are coordinates along the M5-brane and $i, j = 6, 7, 8, 9, 10$ are coordinates transverse to the brane. Additionally, the fermions satisfy appropriate chirality conditions corresponding to the brane breaking half the supersymmetry: $\Gamma_{012345}\epsilon = \epsilon$ and $\Gamma_{012345}\Psi = -\Psi$, and as usual I work with full 32 component spinors. One can recast the equation only in terms of the three-form field strength in which case these equations are equivalent to

$$\begin{aligned} 0 &= \partial^2 X^i \\ 0 &= i\Gamma^\mu \partial_\mu \Psi \\ 0 &= \partial_{[\mu} H_{\nu\rho\lambda]} \\ H_{\mu\nu\rho} &= \frac{1}{3!}\varepsilon_{\mu\nu\rho\sigma\lambda\tau} H_{\sigma\lambda\tau} \end{aligned} \tag{2.2.4}$$

and

$$\begin{aligned} \delta X^i &= i\bar{\epsilon}\Gamma^i \Psi \\ \delta H_{\mu\nu\lambda} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi \\ \delta\Psi &= \partial_\mu X^i \Gamma^\mu \Gamma^i \epsilon + \frac{1}{2\cdot 3!}\Gamma^{\mu\nu\rho} H_{\mu\nu\rho}\epsilon. \end{aligned} \tag{2.2.5}$$

For N M5-branes there exists an interacting CFT in six-dimensions, dubbed the $(2,0)$ theory, that captures their low energy dynamics, decoupled from gravity [132, 142]. This theory can be approached by attempting to construct non-abelian generalisations of the above abelian theory. There are various propositions for this process but a reliable description for multiple M5-branes has thus far proved elusive. In fact it is believed that there is no six-dimensional diffeomorphism invariant action for such a theory [134].

However, the reduction of M5-branes is well known to describe D4-branes; from this perspective indirect studies of the M5-brane can be attempted. Although the $(2,0)$ theory is poorly understood, it is known that when reduced on a circle of radius $R = g_{YM}^2/4\pi^2$ the $(2,0)$ theory reduces to five-dimensional maximally supersymmetric Yang-Mills with gauge group $U(N)$ and coupling g_{YM}, the low energy theory on N D4-branes [143]. Since this theory is perturbatively non-renormalizable the six-dimensional $(2,0)$ CFT provides a UV-completion with an enhanced Lorentz symmetry. It is therefore of great interest to try to understand in detail the relation of the $(2,0)$ theory to five-dimensional maximally supersymmetric Yang-Mills. In particular one would like to know what additional states or degrees of freedom arise in the $(2,0)$ theory that are needed to UV complete five-dimensional maximally supersymmetric Yang-Mills. It has been suggested that all such states are already present in the five-dimensional theory non-perturbatively [144, 143] and that it is in fact well-defined without new degrees of freedom. This discussion will be pertinent in Chapter 5 where M5-branes wrapped on a specific geometry are studied by reducing the six-dimensional abelian theory on S^1, and then finding the non-abelian generalisation of this theory.

There are further uses of six-dimensional $(2,0)$ theories which have been studied. In particular one can take the abelian $(2,0)$ multiplet and construct a non-abelian generalisation by the introduction of various non-dynamical form fields. This setup leads to various interesting systems of M-branes; it will be described in the next section, and forms the basis of Chapters 3 and 4.

2.3 More Brane Systems

In this section I provide an overview of various extensions to the $(2,0)$ theory which lead to novel descriptions of multiple M2 and M5-branes.

2.3.1 A Non-Abelian Extension of $(2,0)$

In [145] a system was presented which generalises the abelian six-dimensional $(2,0)$ multiplet to a non-abelian theory; I will briefly review the construction of this theory below.

The primary goal is to construct a non-abelian generalisation which may admit a description of multiple M5-branes, to this end one starts with the abelian system (2.2.4), and places all fields in a vector space as $X^i = X^i_A T^A$, just as in Section 2.1.1. In order to facilitate this, one also introduces a covariant derivative with gauge field $A_{\mu}{}^B{}_A$ acting on fields as

$$D_\mu X^i_A = \partial_\mu X^i_A - A_{\mu}{}^B{}_A X^i_B. \tag{2.3.1}$$

The argument is then that upon a circle reduction, this theory should reduce to five-dimensional super Yang-Mills theory as discussed above. The supersymmetry transformations in five-dimensional super Yang-Mills contain a term in the fermion variation which goes like

$$\delta\Psi \sim [X^i, X^i]\Gamma^{ij}\Gamma^5\epsilon. \tag{2.3.2}$$

To recover such a term, the non-abelian $(2,0)$ multiplet must have extra terms present which include another gamma matrix to contract with a new field along the brane. It is also clear that the theory will have triple-product like structures. In [145] this is enforced and the demand that the supersymmetry transformations close implies equations of motion and fixes the 3-algebra structure to be the same as discovered in Section 2.1.1. In total this system is then given by

$$\begin{aligned}
0 &= D^2X^i - \frac{i}{2}[Y^\mu, \bar{\Psi}, \Gamma_\mu\Gamma^i\Psi] - [Y^\mu, X^j, [Y_\mu, X^j, X^i]] \\
0 &= D_{[\mu}H_{\nu\lambda\rho]} + \frac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, X^i, D^\tau X^i] + \frac{i}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, \bar{\Psi}, \Gamma^\tau\Psi] \\
0 &= \Gamma^\mu D_\mu\Psi + \Gamma^\mu\Gamma^i[Y_\mu, X^i, \Psi] \\
0 &= F_{\mu\nu}(\cdot) - [Y^\lambda, H_{\mu\nu\lambda}, \cdot] \\
0 &= D_\mu Y^\nu = [Y^\mu, Y^\nu, \cdot] = [Y^\mu, D_\mu(\cdot), \cdot]
\end{aligned} \tag{2.3.3}$$

with supersymmetry transformations

$$\begin{aligned}
\delta X^i &= i\bar{\epsilon}\Gamma^i\Psi \\
\delta Y^\mu &= 0 \\
\delta\Psi &= \Gamma^\mu\Gamma^i D_\mu X^i + \frac{1}{2\cdot 3!}H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\epsilon - \frac{1}{2}\Gamma_\mu\Gamma^{ij}[Y^\mu, X^i, X^j]\epsilon \\
\delta H_{\mu\nu\rho} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}D_{\rho]}\Psi + i\bar{\epsilon}\Gamma^i\Gamma_{\mu\nu\rho\lambda}[Y^\lambda, X^i, \Psi] \\
\delta A_\mu(\cdot) &= i\bar{\epsilon}\Gamma_{\mu\nu}[Y^\nu, \Psi, \cdot].
\end{aligned} \tag{2.3.4}$$

In this system Y^μ is the new auxilliary field which is added to enforce the correct reduction. To get the correct scaling one finds that this field has scaling dimension $[Y] = -1$ so should be considered as a length. I note also that the spinors satisfy the same (anti)-chirality relations as described in the abelian $(2,0)$ multiplet above.

The system has various interpretations in terms of branes depending on how one turns on the Y field. For a simple choice of structure constant and metric for the 3-algebra, $f^{ABCD} = \frac{2\pi}{k}\varepsilon^{ABCD}$ and $h^{AB} = \delta^{AB}$ then Y_μ can be fixed to lie only in one direction on the 3-algebra and so $Y^\mu_A = V^\mu\delta^4_A$. From here the system reduces considerably, all the triple-products contain this field so they either reduce naturally to Lie brackets or they vanish. In the latter case, where one takes the $A = 4$ component of each equation, one recovers a decoupled abelian six-dimensional $(2,0)$ tensor multiplet corresponding to a centre of mass.

When $A \neq 4$ consider that the constraints found in (2.3.3) imply that all fields must be independent of those coordinates which lie parallel to V^μ—this is why we should consider V^μ turned on only in particular directions. There are three cases of interest discussed in [145].

Firstly, if $V^5 \neq 0$ only then one reduces simply to five-dimensional maximally supersymmetric Yang-Mills theory with gauge group $SU(2)$; this makes sense for a pair of M5-branes reducing to a pair of D4-branes. If the choice is instead timelike, $V^0 \neq 0$ only, then the same system is recovered except with euclidean signature. The final case is a null direction, $V^+ \neq 0$ only (see the appendix for a suitable definition of a null x^+ coordinate). This case was described fully in [146] and will be seen again in Section 4.3. Under such a choice one finds that the self-duality of the three-form implies that the field strength, defined only on the remaining four spacelike directions, is self dual. Therefore one can use the ADHM construction to solve for the remaining fields on the instanton moduli space. In the construction x^- takes the role of time, quantisation can occur, and the system reduces completely to quantum mechanics acting on the instanton moduli space [146]. These systems are all of interest for describing multiple M5-branes.

2.3.2 A Further Extension

One can generalise this theory further. In [147] an extension was considered where an abelian three-form was added to the theory. This particular addition was chosen because, by analogy with the system of Section 2.3.1, the three-form will fix three non-dynamical directions in the theory; this leaves three remaining directions which makes it a good candidate for describing systems of M2-branes. This will be realised below.

The conventions are exactly as in Section 2.3.1 and I introduce an abelian three-form, $C_{\mu\nu\rho}$, into the algebra as described in [147]. One demands closure of this new extended supersymmetry algebra which turns out to be very prescriptive, the equations of motion are found to be

$$
\begin{aligned}
0 &= D^2 X^i - \frac{i}{2}[Y^\sigma, \bar{\Psi}, \Gamma_\sigma \Gamma^i \Psi] + [Y^\sigma, X^j, [Y_\sigma, X^j, X^i]] \\
&\quad + \frac{i}{2\cdot 3!} C^{\sigma\tau\omega}[\bar{\Psi}, \Gamma_{\sigma\tau\omega}\Gamma^{ij}\Psi, X^j] + \frac{1}{2\cdot 3!} C^{\sigma\tau\omega} C_{\sigma\tau\omega}[[X^i, X^j, X^k], X^j, X^k] \\
0 &= D_{[\lambda} H_{\mu\nu\rho]} + \frac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, X^i, D^\tau X^i] - \frac{1}{2}(\star C)_{[\mu\nu\lambda}[X^i, X^j, [Y_{\rho]}, X^i, X^j]] \\
&\quad + \frac{i}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, \bar{\Psi}, \Gamma^\tau \Psi] - \frac{i}{2}(\star C)_{[\mu\nu\lambda}[X^i, \bar{\Psi}, \Gamma_{\rho]}\Gamma^i \Psi] \\
0 &= \Gamma^\rho D_\rho \Psi + \Gamma_\rho \Gamma^i [Y^\rho, X^i, \Psi] + \frac{1}{2\cdot 3!}\Gamma_{\rho\sigma\tau} C^{\rho\sigma\tau}\Gamma^{ij}[X^i, X^j, \Psi] \ ,
\end{aligned} \tag{2.3.5}
$$

with the following constraint equations

$$
\begin{aligned}
0 &= F_{\mu\nu}(\cdot) - [Y^\lambda, H_{\mu\nu\lambda}, \ \cdot\ ] + (\star C)_{\mu\nu\lambda}[X^i, D^\lambda X^i, \ \cdot\ ] + \frac{i}{2}(\star C)_{\mu\nu\lambda}[\bar{\Psi}, \Gamma^\lambda \Psi, \ \cdot\ ] \\
0 &= D_\nu Y^\mu - \frac{1}{2} C^{\mu\lambda\rho} H_{\nu\lambda\rho} \\
0 &= C^{\mu\nu\sigma} D_\sigma(\cdot) + \ [Y^\mu, Y^\nu, \ \cdot\ ] \\
0 &= [Y^\nu, D_\nu \cdot\ , \ \cdot' \ ] + \frac{1}{3!} C^{\sigma\tau\omega}[H_{\sigma\tau\omega}, \ \cdot\ , \ \cdot' \ ] \\
0 &= C \wedge Y \\
0 &= C_{[\mu\nu}{}^\rho C_{\lambda]\rho}{}^\sigma \ .
\end{aligned}
\tag{2.3.6}
$$

The three-form $H_{\mu\nu\rho}$ is of course still required to be self-dual

$$
H_{\mu\nu\rho} = \frac{1}{3!}\varepsilon^{\mu\nu\rho\sigma\lambda\tau} H_{\rho\sigma\lambda}. \tag{2.3.7}
$$

The supersymmetry transformations are given by

$$
\begin{aligned}
\delta X^i &= i\bar{\epsilon}\Gamma^i \Psi \\
\delta Y^\mu &= \frac{i}{2}\bar{\epsilon}\Gamma_{\lambda\rho} C^{\mu\lambda\rho}\Psi \\
\delta \Psi &= \Gamma^\mu \Gamma^i D_\mu X^i \epsilon + \frac{1}{2\cdot 3!} H_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\epsilon \\
&\quad - \frac{1}{2}\Gamma_\mu \Gamma^{ij}[Y^\mu, X^i, X^j]\epsilon + \frac{1}{3!^2} C_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\Gamma^{ijk}[X^i, X^j, X^k]\epsilon \\
\delta H_{\mu\nu\lambda} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu} D_{\lambda]}\Psi + i\bar{\epsilon}\Gamma^i \Gamma_{\mu\nu\lambda\rho}[Y^\rho, X^i, \Psi] \\
&\quad + \frac{i}{2}\bar{\epsilon}(\star C)_{\mu\nu\lambda}\Gamma^{ij}[X^i, X^j, \Psi] + \frac{3i}{4}\bar{\epsilon}\Gamma_{[\mu\nu|\rho\sigma} C^{\rho\sigma}{}_{\lambda]}\Gamma^{ij}[X^i, X^j, \Psi] \\
\delta A_\mu(\cdot) &= i\bar{\epsilon}\Gamma_{\mu\nu}[Y^\nu, \Psi, \ \cdot\ ] - \frac{i}{3!}\bar{\epsilon} C^{\nu\lambda\rho}\Gamma_{\mu\nu\lambda\rho}\Gamma^i[X^i, \Psi, \ \cdot\ ] \ .
\end{aligned}
\tag{2.3.8}
$$

One can check that for $C = 0$ this system reduces to that in Section 2.3.1. In addition demanding appropriate scaling dimensions in the theory implies that C has dimension $[C] = -3$.

In the same way as before, one can now consider what happens when different components of the three-form are turned on. There are two choices discussed in [147]. The first choice is $C_{012} = l^3$ only, for some length l. In this case one sees manifestly that the $SO(1,5)$ symmetry breaks to $SO(1,2) \times SO(3)$, however it turns out that this new $SO(3)$ then enhances with the five transverse coordinates to an $SO(8)$, so that the total symmetry of the theory is given by $SO(1,2) \times SO(8)$. This is precisely as expected for M2-branes; in fact it can be shown that the model which this choice lands on is the BLG model of Section 2.1.1.

Another choice to make is that of $C_{345} = l^3$ only. This comes from a wick rotation of the above and so describes a euclidean three-dimensional theory which can be shown to contain a $SO(2,6)$ R-symmetry. This is similar to as found in [148] in describing euclidean M2-branes.

This concludes the discussions of this chapter. I have presented a summary of key systems and ideas in our understanding of non-abelian M2 and M5-branes. The following chapters will use the machinery developed here to explore various novel results which constitute the original research conducted during my studies.

Chapter 3

Extended Systems of Null M2-branes

In this chapter I extend the background material of Chapter 2 to describe a novel system describing multiple null M-branes. What is meant by null will be discussed later, but essentially it is simply a statement that one of the directions along the brane is taken to have undergone an infinite boost. This work was published in JHEP [9] with Neil Lambert and Piotr Kucharski.

3.1 Introduction

As described in Section 2.3, a closed system of equations for various six-dimensional fields was found in [145, 147] that are invariant under the $(2, 0)$ superalgebra which is associated to the worldvolume of M5-branes embedded in an eleven-dimensional spacetime. The fields take values in a 3-algebra, except for the gauge field that takes values in the Lie-algebra (specifically $su(2) \oplus su(2)$ for the case at hand) that acts on the 3-algebra. The system can be thought of as a set of dynamical equations for the scalars, fermions and self-dual three-form as well as constraints for the additional gauge and vector fields that it contains. In addition the system depends on a choice of abelian three-form $C_{\mu\nu\lambda}$. For $C_{\mu\nu\lambda} = 0$ it reproduces various descriptions of two M5-branes [145, 146, 149]. For $C_{\mu\nu\lambda}$ spacelike the constraints reduce it to the equations for two M2-branes [147]. In this chapter I will explore the system for a null choice of $C_{\mu\nu\lambda}$. It will be seen that this leads to a novel supersymmetric system of equations on $\mathbb{R}^2$ times a null direction $\mathbb{R}_+$. Alternatively, via an M-theory version of T-duality, we can think of this system as describing intersecting M2-branes which are tangent to a null direction.

A similar system of equations, but defined on $\mathbb{R}^4$ times a null direction $\mathbb{R}_+$, was obtained in [145] (and is therefore also a solution to the constraints of [147]). These were analysed in [146] where it was shown they reduce to dynamics on the instanton moduli space with the null direction playing the role of 'time'. From the origin of these equations in the $(2,0)$ superalgebra it is clear that the resulting system describes two M5-branes compactified on a null circle with corresponding null momentum given by the instanton number. This is in agreement with the DLCQ prescription of [150, 151]. One similarly expects that the system described in this section corresponds to two M5-branes compactified on $\mathbb{T}^2$ and carrying momentum along the null direction. I show that the system reduces to quantum mechanics on the Hitchin moduli space and provides a description of intersecting null M2-branes. There is a similar DLCQ description of four-dimensional maximally supersymmetric $SU(N)$ Yang-Mills with null momentum K which is also based on quantum mechanics on Hitchin moduli space [152, 153]. It will be argued that this construction is related to the system presented here by U-duality.

The rest of this chapter is organised as follows. In Section 3.2 I work from the system of [147], which was described in Section 2.3.2 and examine it for the case of a null background three-form C_3. In Section 3.3 this new system is analysed and in particular it is shown how, for a particular choice of fields, the system reduces to supersymmetric dynamics on the moduli space of solutions to Hitchin's equations. In Section 3.4 I provide a physical interpretation of the system in terms of intersecting M2-branes. Section 3.5 contains a summary and comments on the results. Conventions used throughout this chapter, where not given in the text, can be found in the appendix.

3.2 The System

In this chapter I work from the system described in section 2.3.2 with the following choice of the background three-form, C:

$$C_{34+} = l^3, \tag{3.2.1}$$

where

$$x^+ = \frac{x^5 + x^0}{\sqrt{2}} \qquad x^- = \frac{x^5 - x^0}{\sqrt{2}} \ . \tag{3.2.2}$$

In particular I will show that the solution of the constraints leads to fields that only depend on x^+, x^1, x^2. Although the system initially has an $SO_L(1,5) \times SO_R(5)$ symmetry turning on C_{+34} breaks the Lorentz group $SO_L(1,5)$ to an $SO_L(2)$ that acts as rotations in the (x^1, x^2)-plane along with an $SO_R(2)$ that acts as rotations in the (x^3, x^4)-plane and which is now viewed as an R-symmetry. Somewhat surprisingly one finds that there is an enhancement of the original $SO_R(5)$ R-symmetry to $SO_R(6)$ so that the final system has an $SO_L(2) \times SO_R(2) \times SO_R(6)$ symmetry.

To exhibit this symmetry on the fermions it is useful to introduce a new representation of the $Spin(1,10)$ Clifford algebra:

$$\begin{aligned}
\hat{\Gamma}_0 &= \Gamma_{0534} \\
\hat{\Gamma}_{1,2} &= \Gamma_0 \Gamma_{1,2} \\
\hat{\Gamma}_{3,4} &= \Gamma_{05} \Gamma_{4,3} \\
\hat{\Gamma}^5 &= \Gamma_0 \Gamma_{34} \\
\hat{\Gamma}^i &= \Gamma_0 \Gamma^i \ ,
\end{aligned}$$

which satisfy $\{\hat{\Gamma}_m, \hat{\Gamma}_n\} = 2\eta_{mn}$, $m, n = 0, 1, 2, .., 10$. However in what follows I will only be interested in the $Spin(10)$ subalgebra which is broken to $Spin(2) \times Spin(2) \times Spin(6)$. I also decompose any spinor χ as $\chi = \chi_+ + \chi_-$ where

$$\Gamma_{05}\chi_\pm = \hat{\Gamma}_{034}\chi_\pm = \pm\chi_\pm \ . \tag{3.2.3}$$

3.2.1 Solving the Constraints and Equations of Motion

The first task is to solve the constraints. From the last constraint in (2.3.6) one sees that only Y^-, Y^3, Y^4 are non-vanishing. The third and fourth equations in (2.3.6) can be reduced to algebraic equations if one takes $\partial_-, \partial_3, \partial_4$ to vanish. Thus all fields are functions of x^+, x^1, x^2. Solving the resulting algebraic equations from the third and fourth equations in (2.3.6) one finds that

$$
\begin{aligned}
A_{-} &= \frac{1}{l^3}\left[Y^3, Y^4, \cdot\right] \\
A_3 &= \frac{1}{l^3}\left[Y^4, Y^{-}, \cdot\right] \\
A_4 &= -\frac{1}{l^3}\left[Y^3, Y^{-}, \cdot\right] \ .
\end{aligned} \tag{3.2.4}
$$

Next one can use the second equation in (2.3.6) to determine the components of $H_{\mu\nu\lambda}$. Using self-duality one finds

$$
\begin{aligned}
H_{34-} = H_{12-} &= -\frac{1}{l^6}\left[Y^3, Y^4, Y^{-}\right] \\
H_{34+} = -H_{12+} &= \frac{1}{l^3} D_{+} Y^{-} \\
H_{3-+} = H_{124} &= -\frac{1}{l^3} D_{+} Y^4 \\
H_{4-+} = -H_{123} &= \frac{1}{l^3} D_{+} Y^3 \\
H_{134} = -H_{2-+} &= \frac{1}{l^3} D_1 Y^{-} \\
H_{234} = H_{1-+} &= \frac{1}{l^3} D_2 Y^{-} \\
-\frac{1}{l^3} D_1 Y^4 = H_{13-} = -H_{24-} &= -\frac{1}{l^3} D_2 Y^3 \\
\frac{1}{l^3} D_1 Y^3 = H_{14-} = H_{23-} &= -\frac{1}{l^3} D_2 Y^4 \ .
\end{aligned} \tag{3.2.5}
$$

To proceed it is useful to introduce the complex coordinates and fields

$$
\begin{aligned}
z &= x^1 + i x^2 \qquad \bar{z} = x^1 - i x^2 \\
Z &= Y^4 + i Y^3 \qquad \bar{Z} = Y^4 - i Y^3 \ .
\end{aligned} \tag{3.2.6}
$$

Here, and in what follows, a bar denotes complex conjugation and not the Dirac conjugate. In addition I introduce an $SO(6)$ multiplet of scalar fields X^I, $I = 5, 6, ..., 10$, defined by

$$
X^5 = l^{-3} Y^{-} \qquad X^I = X^i \ I = 6, ..., 10 \ . \tag{3.2.7}
$$

First note that there is one independent component of $H_{\mu\nu\lambda}$ that is not determined from the constraints above and so I define

$$H = H_{+z3} = iH_{+z4}\ . \tag{3.2.8}$$

One then finds that the self-dual conditions $H_{13-} = -H_{24-}$ and $H_{14-} = H_{23-}$ are equivalent to

$$\bar{D}Z = 0\ . \tag{3.2.9}$$

The remaining constraints can now be evaluated to give

$$\begin{aligned} F_{+z}(\cdot) &= il^3\left[X^I, DX^I, \cdot\right] - i\left[Z, H,\ \cdot\ \right] - \frac{l^3}{2}\left(\left[\Psi_+^T, \hat{\Gamma}_z\Psi_-, \cdot\right] + \left[\Psi_-^T, \hat{\Gamma}_z\Psi_+, \cdot\right]\right) \\ F_{z\bar{z}}(\cdot) &= -\frac{i}{4l^3}\left(\left[Z, D_+\bar{Z},\ \cdot\ \right] + \left[\bar{Z}, D_+Z, \cdot\right]\right) - \frac{1}{4}\left[X^I, \left[Z, \bar{Z}, X^I\right],\ \cdot\ \right] - \frac{l^3}{2\sqrt{2}}\left[\Psi_+^T, \Psi_+, \cdot\right]\ . \end{aligned} \tag{3.2.10}$$

The last job is to evaluate the equations of motion. The scalar equation becomes

$$\begin{aligned} 0 = &\ 2(D\bar{D} + \bar{D}D)X^I + \frac{i}{2l^3}[D_+Z, \bar{Z}, X^I] + \frac{i}{2l^3}[Z, D_+\bar{Z}, X^I] + \frac{i}{l^3}[Z, \bar{Z}, D_+X^I] \\ &+ \frac{1}{2}\left[Z, X^J, [\bar{Z}, X^J, X^I]\right] + \frac{1}{2}\left[\bar{Z}, X^J, [Z, X^J, X^I]\right] - l^3\sqrt{2}\left[\Psi_+^T, \hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^{IJ}\Psi_+, X^J\right] \\ &+ \frac{i}{2}\left(\left[Z, \Psi_+^T, \hat{\Gamma}_Z\hat{\Gamma}^I\Psi_-\right] - \left[Z, \Psi_-^T, \hat{\Gamma}_Z\hat{\Gamma}^I\Psi_+\right] + \left[\bar{Z}, \Psi_+^T, \hat{\Gamma}_{\bar{Z}}\hat{\Gamma}^I\Psi_-\right] - \left[\bar{Z}, \Psi_-^T, \hat{\Gamma}_{\bar{Z}}\hat{\Gamma}^I\Psi_+\right]\right), \end{aligned} \tag{3.2.11}$$

where the $I = 5$ component actually arises from the $(DH)_{z\bar{z}+-}$ equation. The only other new equation that arises from the $(DH)_{\mu\nu\lambda}$ equation comes from the $(DH)_{z\bar{z}+3}$ and $(DH)_{z\bar{z}+4}$ terms and gives

$$\begin{aligned} 0 = &\ D_+^2 Z + il^3[Z, X^I, D_+X^I] - \frac{l^6}{2}[X^I, X^J, [X^I, X^J, Z]] + 4l^3 D\bar{H} \\ &+ \frac{l^3}{\sqrt{2}}\left[Z, \Psi_-^T, \Psi_-\right] + il^6\left(\left[\Psi_+^T, \hat{\Gamma}_{\bar{Z}}\hat{\Gamma}^I\Psi_-, X^I\right] - \left[\Psi_-^T, \hat{\Gamma}_{\bar{Z}}\hat{\Gamma}^I\Psi_+, X^I\right]\right), \end{aligned} \tag{3.2.12}$$

The fermion equations are

$$\begin{aligned} 0 = &\ D_+\Psi_+ + \sqrt{2}\hat{\Gamma}_z\bar{D}\Psi_- + \sqrt{2}\hat{\Gamma}_{\bar{z}}D\Psi_- + il^3\hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^{IJ}\left[X^I, X^J, \Psi_+\right] \\ &+ \frac{1}{\sqrt{2}}\hat{\Gamma}^I\hat{\Gamma}_Z\left[Z, X^I, \Psi_-\right] + \frac{1}{\sqrt{2}}\hat{\Gamma}^I\hat{\Gamma}_{\bar{Z}}\left[\bar{Z}, X^I, \Psi_-\right]. \end{aligned} \tag{3.2.13}$$

and

$$\begin{aligned}0 = &\sqrt{2}\hat{\Gamma}_z \bar{D}\Psi_+ + \sqrt{2}\hat{\Gamma}_{\bar{z}} D\Psi_+ - \frac{i}{2l^3}\left[Z, \bar{Z}, \Psi_-\right] \\ &- \frac{1}{\sqrt{2}}\hat{\Gamma}^I\hat{\Gamma}_Z\left[Z, X^I, \Psi_+\right] - \frac{1}{\sqrt{2}}\hat{\Gamma}^I\hat{\Gamma}_{\bar{Z}}\left[\bar{Z}, X^I, \Psi_+\right]. \end{aligned} \tag{3.2.14}$$

Here one sees that the equations of motion have a natural $SO_L(2)\times SO_R(2)\times SO_R(6)$ symmetry. In particular the field Y^- has enhanced the original $SO_R(5)$ to $SO_R(6)$.

3.2.2 Supersymmetry

The supersymmetry transformations can also be expressed as

$$\begin{aligned}\delta X^I =& i\epsilon_+^T\hat{\Gamma}^I\Psi_- + i\epsilon_-^T\hat{\Gamma}^I\Psi_+ \\ \delta Z =& 2\sqrt{2}l^3\epsilon_+^T\hat{\Gamma}_{\bar{Z}}\Psi_+ \\ \delta \bar{Z} =& -2\sqrt{2}l^3\epsilon_+^T\hat{\Gamma}_Z\Psi_+ \\ \delta A_z =& \sqrt{2}l^3\epsilon_+^T\hat{\Gamma}^I\hat{\Gamma}_z[X^I, \Psi_+, \cdot] + i\epsilon_-^T\hat{\Gamma}_z\hat{\Gamma}_{\bar{Z}}\left[\bar{Z}, \Psi_+, \cdot\right] - i\epsilon_+^T\hat{\Gamma}_z\hat{\Gamma}_Z\left[Z, \Psi_-, \cdot\right] \\ \delta A_{\bar{z}} =& -\sqrt{2}l^3\epsilon_+^T\hat{\Gamma}^I\hat{\Gamma}_{\bar{z}}[X^I, \Psi_+, \cdot] + i\epsilon_-^T\hat{\Gamma}_{\bar{z}}\hat{\Gamma}_Z\left[Z, \Psi_+, \cdot\right] - i\epsilon_+^T\hat{\Gamma}_{\bar{z}}\hat{\Gamma}_{\bar{Z}}\left[\bar{Z}, \Psi_-, \cdot\right] \\ \delta A_+ =& \sqrt{2}i\epsilon_-^T\hat{\Gamma}_Z\left[Z, \Psi_-, \cdot\right] + \sqrt{2}i\epsilon_-^T\hat{\Gamma}_{\bar{Z}}\left[\bar{Z}, \Psi_-, \cdot\right] \\ &+ 2l^3\epsilon_-^T\hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^I\left[X^I, \Psi_+, \cdot\right] - 2l^3\epsilon_+^T\hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^I\left[X^I, \Psi_-, \cdot\right] \end{aligned} \tag{3.2.15}$$

for the bosons, and as

$$
\begin{aligned}
\delta\Psi_+ =& \frac{i}{\sqrt{2}l^3}\hat{\Gamma}^I\left[Z,\bar{Z},X^I\right]\epsilon_- - \frac{i}{l^3}\left(\hat{\Gamma}_Z D_+ Z - \hat{\Gamma}_{\bar{Z}} D_+\bar{Z}\right)\epsilon_+ \\
& - \frac{1}{2}\left(\hat{\Gamma}_Z\hat{\Gamma}^{IJ}\left[Z,X^I,X^J\right] + \hat{\Gamma}_{\bar{Z}}\hat{\Gamma}^{IJ}\left[\bar{Z},X^I,X^J\right]\right)\epsilon_+ \\
& + 2\left(\hat{\Gamma}_{\bar{z}}\hat{\Gamma}^I D X^I + \hat{\Gamma}_z\hat{\Gamma}^I\bar{D}X^I\right)\epsilon_+ \\
& + \frac{\sqrt{2}i}{l^3}\left(\hat{\Gamma}_{\bar{z}}\hat{\Gamma}_Z D Z - \hat{\Gamma}_z\hat{\Gamma}_{\bar{Z}}\bar{D}\bar{Z}\right)\epsilon_- \\
\delta\Psi_- =& -\sqrt{2}\hat{\Gamma}^I D_+ X^I\epsilon_+ - \frac{\sqrt{2}il^3}{3}\hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^{IJK}\left[X^I,X^J,X^K\right]\epsilon_+ \\
& + \frac{1}{2}\left(\hat{\Gamma}_Z\hat{\Gamma}^{IJ}\left[Z,X^I,X^J\right] + \hat{\Gamma}_{\bar{Z}}\hat{\Gamma}^{IJ}\left[\bar{Z},X^I,X^J\right]\right)\epsilon_- \\
& - \frac{i}{l^3}\left(\hat{\Gamma}_Z D_+ Z - \hat{\Gamma}_{\bar{Z}} D_+\bar{Z}\right)\epsilon_- \\
& + 2\left(\hat{\Gamma}_{\bar{z}}\hat{\Gamma}^I D X^I + \hat{\Gamma}_z\hat{\Gamma}^I\bar{D}X^I\right)\epsilon_- \\
& + 2\sqrt{2}i\left(\hat{\Gamma}_{\bar{z}}\hat{\Gamma}_{\bar{Z}} H - \hat{\Gamma}_z\hat{\Gamma}_Z\bar{H}\right)\epsilon_+
\end{aligned}
\tag{3.2.16}
$$

for the fermions. The variation of $H = H_{+z3}$ requires special attention as self-duality implies that $H = iH_{+z4}$. Evaluating these gives

$$
\begin{aligned}
\delta H_{+z3} =& \sqrt{2}\epsilon_-^T\left(\hat{\Gamma}_Z - \hat{\Gamma}_{\bar{Z}}\right)D\Psi_- + \epsilon_+^T\hat{\Gamma}_z\hat{\Gamma}_Z D_+\Psi_- + \epsilon_-^T\hat{\Gamma}_z\hat{\Gamma}_{\bar{Z}} D_+\Psi_+ \\
& + \frac{i}{2}l^3\epsilon_-^T\hat{\Gamma}_z\hat{\Gamma}_{\bar{Z}}\hat{\Gamma}^{IJ}\left[X^I,X^J,\Psi_+\right] + \frac{i}{2}l^3\epsilon_+^T\hat{\Gamma}_z\hat{\Gamma}_Z\hat{\Gamma}^{IJ}\left[X^I,X^J,\Psi_-\right] \\
& + \sqrt{2}\epsilon_-^T\hat{\Gamma}_z\hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^I[Z+\bar{Z},X^I,\Psi_-] \\
i\delta H_{+z4} =& \sqrt{2}\epsilon_-^T\left(\hat{\Gamma}_Z + \hat{\Gamma}_{\bar{Z}}\right)D\Psi_- + \epsilon_+^T\hat{\Gamma}_z\hat{\Gamma}_Z D_+\Psi_- - \epsilon_-^T\hat{\Gamma}_z\hat{\Gamma}_{\bar{Z}} D_+\Psi_+ \\
& - \frac{i}{2}l^3\epsilon_-^T\hat{\Gamma}_z\hat{\Gamma}_{\bar{Z}}\hat{\Gamma}^{IJ}\left[X^I,X^J,\Psi_+\right] + \frac{i}{2}l^3\epsilon_+^T\hat{\Gamma}_z\hat{\Gamma}_Z\hat{\Gamma}^{IJ}\left[X^I,X^J,\Psi_-\right] \\
& - \sqrt{2}\epsilon_-^T\hat{\Gamma}_z\hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^I[Z-\bar{Z},X^I,\Psi_-]\ .
\end{aligned}
\tag{3.2.17}
$$

Demanding that these are equal gives the condition

$$
\epsilon_-^T\left(\sqrt{2}\hat{\Gamma}_{\bar{Z}}D\Psi_- - \hat{\Gamma}_z\hat{\Gamma}_{\bar{Z}}D_+\Psi_+ - \frac{i}{2}l^3\hat{\Gamma}_z\hat{\Gamma}_{\bar{Z}}\hat{\Gamma}^{IJ}\left[X^I,X^J,\Psi_+\right] + \sqrt{2}\hat{\Gamma}_z\hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^I[Z,X^I,\Psi_-]\right) = 0
\tag{3.2.18}
$$

As required this vanishes as a consequence of the fermion equation (3.2.12). As a result I find

$$\begin{aligned}\delta H =&\sqrt{2}\epsilon_-^T\hat{\Gamma}_Z D\Psi_- + \epsilon_+^T\hat{\Gamma}_z\hat{\Gamma}_Z D_+\Psi_-\\ &+\frac{i}{2}l^3\epsilon_+^T\hat{\Gamma}_z\hat{\Gamma}_Z\hat{\Gamma}^{IJ}\left[X^I,X^J,\Psi_-\right]+\sqrt{2}\epsilon_-^T\hat{\Gamma}_z\hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^I[\bar{Z},X^I,\Psi_-]\ .\end{aligned} \tag{3.2.19}$$

It is worth commenting that the identification $H_{+z3} = iH_{+z4}$ maps the $SO_R(2)$ action as rotation by θ on x^3, x^4 to the $U(1)$ action $H \to e^{i\theta}H$.

I also note that a rescaling of l can be absorbed by a rescaling of x^+ and H. Henceforth I simply take $l = 1$.

3.2.3 Energy-Momentum and Superalgebra

The general form for the supercurrent and energy-momentum tensor were given in [147] as:

$$\begin{aligned}S^\mu =& -2\pi i\langle D_\nu X^i,\Gamma^\nu\Gamma^i\Gamma^\mu\Psi\rangle+\frac{\pi i}{3!}\langle H_{\sigma\tau\omega},\Gamma^{\sigma\tau\omega}\Gamma^\mu\Psi\rangle-\pi i\langle[Y_\nu,X^i,X^j],\Gamma^\nu\Gamma^{ij}\Gamma^\mu\Psi\rangle\\ &+\frac{\pi i}{3\cdot 3!}C_{\sigma\tau\omega}\langle[X^i,X^j,X^k],\Gamma^{ijk}\Gamma^{\sigma\tau\omega}\Gamma^\mu\Psi\rangle\ ,\end{aligned} \tag{3.2.20}$$

and[1]

$$\begin{aligned}T_{\mu\nu} =&2\pi\langle D_\mu X^i,D_\nu X^i\rangle-\pi\eta_{\mu\nu}\langle D_\lambda X^i,D^\lambda X^i\rangle+\pi\langle[X^i,X^j,Y_\mu],[X^i,X^j,Y_\nu]\rangle\\ &-\frac{\pi}{2}\eta_{\mu\nu}\langle[X^i,X^j,Y_\lambda],[X^i,X^j,Y^\lambda]\rangle+\frac{\pi}{2}\langle H_{\mu\lambda\rho},H_\nu{}^{\lambda\rho}\rangle\\ &-i\pi\langle\bar{\Psi},\Gamma_\mu D_\nu\Psi\rangle+i\pi\eta_{\mu\nu}\langle\bar{\Psi},\Gamma^\lambda D_\lambda\Psi\rangle-i\pi\eta_{\mu\nu}\langle[\bar{\Psi},Y^\lambda,X^i],\Gamma_\lambda\Gamma^i\Psi\rangle\\ &+\frac{\pi}{3!}\langle[X^i,X^j,X^k],[X^i,X^j,X^k]\rangle(C_{\mu\tau\omega}C_\nu{}^{\tau\omega}-\frac{1}{3!}\eta_{\mu\nu}C^2)\\ &+\frac{\pi}{3!}C_{\mu\lambda\rho}(\star C)_\nu{}^{\lambda\rho}\langle[X^i,X^j,X^k],[X^i,X^j,X^k]\rangle-\frac{i\pi}{3!}\eta_{\mu\nu}C^{\sigma\tau\omega}\langle[\bar{\Psi},\Gamma_{\sigma\tau\omega}\Gamma^{ij}\Psi,X^i],X^j\rangle\ .\end{aligned} \tag{3.2.21}$$

[1] This corrects a misprint in the fermion kinetic term contribution to $T_{\mu\nu}$ that appears in [147].

Setting the fermions to zero I find that in the case at hand

$$\begin{aligned}
T_{--} &= 2\pi\langle DZ, \bar{D}\bar{Z}\rangle - \frac{\pi}{2}\langle [Z,\bar{Z},X^I],[Z,\bar{Z},X^I]\rangle \\
&= \pi\partial\langle Z,\bar{D}\bar{Z}\rangle + \pi\bar{\partial}\langle \bar{Z}, DZ\rangle \\
T_{-+} &= -4\pi\langle DX^I,\bar{D}\bar{X}^I\rangle - \frac{\pi}{2}\langle [Z,X^I,X^J],[\bar{Z},X^I,X^J]\rangle - \pi\langle D_+Z, D_+\bar{Z}\rangle \\
&= -2\pi\partial\left(\langle X^I,\bar{D}X^I\rangle + \langle \bar{Z},\bar{H}\rangle\right) - 2\pi\bar{\partial}\left(\langle X^I, DX^I\rangle + \langle Z,H\rangle\right) \\
&\quad - \frac{\pi}{2}\partial_+\left(\langle Z, D_+\bar{Z}\rangle + \langle\bar{Z},D_+Z\rangle\right) \\
T_{-z} &= -\pi\partial\langle Z, D_+\bar{Z}\rangle \ .
\end{aligned} \tag{3.2.22}$$

In the system here the role of time is played by x^+ so define

$$\begin{aligned}
\mathcal{P}_+ &= V_3\int dz d\bar{z}\ T_{-+} \\
\mathcal{P}_z &= V_3\int dz d\bar{z}\ T_{-z} \\
\mathcal{Q}_\pm &= V_3\int dz d\bar{z}\ S^+_\pm \ ,
\end{aligned} \tag{3.2.23}$$

as well as the topological term

$$\mathcal{W} = V_3\int dz d\bar{z}\ T_{--} \ . \tag{3.2.24}$$

Here V_3 is a three-dimensional volume factor that arises from the fact that $T_{\mu\nu}$, as defined above, has dimension six as appropriate for a six-dimensional theory. Given that there is only one length scale in the system it is natural to take $V_3 = l^3$.

After some calculations one finds that the superalgebra takes the form

$$
\begin{aligned}
\{\mathcal{Q}_-,\mathcal{Q}_-\} =& 2\sqrt{2}\mathcal{W}\\
\{\mathcal{Q}_+,\mathcal{Q}_-\} =& -4\mathcal{P}_z\hat{\Gamma}_{\bar{z}}-4\mathcal{P}_{\bar{z}}\hat{\Gamma}_z\\
&+4\mathcal{Z}^I_Z\hat{\Gamma}_{\bar{Z}}\hat{\Gamma}^I+4\mathcal{Z}^I_{\bar{Z}}\hat{\Gamma}_Z\hat{\Gamma}^I\\
&+\frac{1}{2!}\mathcal{Z}^{IJ}_{\bar{z}}\hat{\Gamma}_z\hat{\Gamma}^{IJ}+\frac{1}{2!}\mathcal{Z}^{IJ}_{z}\hat{\Gamma}_{\bar{z}}\hat{\Gamma}^{IJ}\\
&+\frac{1}{3!}\mathcal{Z}^{IJK}_{\bar{Z}}\hat{\Gamma}_Z\hat{\Gamma}^{IJK}+\frac{1}{3!}\mathcal{Z}^{IJK}_{Z}\hat{\Gamma}_{\bar{Z}}\hat{\Gamma}^{IJK}\\
\{\mathcal{Q}_+,\mathcal{Q}_+\} =& -2\sqrt{2}\mathcal{P}_+\\
&+\frac{1}{2!}\mathcal{Z}^{IJ}_{z\bar{z}}\hat{\Gamma}_{z\bar{z}}\hat{\Gamma}^{IJ}+\frac{1}{2!}\mathcal{Z}^{IJ}_{Z\bar{Z}}\hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^{IJ}\\
&+\frac{1}{3!}\mathcal{Z}^{IJK}_{\bar{Z}z}\hat{\Gamma}_{Z\bar{z}}\hat{\Gamma}^{IJK}+\frac{1}{3!}\mathcal{Z}^{IJK}_{\bar{Z}\bar{z}}\hat{\Gamma}_{Zz}\hat{\Gamma}^{IJK}\\
&+\frac{1}{3!}\mathcal{Z}^{IJK}_{Zz}\hat{\Gamma}_{\bar{Z}\bar{z}}\hat{\Gamma}^{IJK}+\frac{1}{3!}\mathcal{Z}^{IJK}_{Z\bar{z}}\hat{\Gamma}_{\bar{Z}z}\hat{\Gamma}^{IJK}+\frac{1}{4!}\mathcal{Z}^{IJKL}\hat{\Gamma}^{IJKL}\ .
\end{aligned}
\tag{3.2.25}
$$

The central charges are given by

$$
\begin{aligned}
\mathcal{Z}^I_Z =&\; 2\pi i V_3\int dzd\bar{z}\partial\langle X^I,\bar{D}\bar{Z}\rangle\\
\mathcal{Z}^I_{\bar{Z}} =&-2\pi i V_3\int dzd\bar{z}\bar{\partial}\langle X^I,DZ\rangle\\
\mathcal{Z}^{IJ}_{\bar{z}} =&\; 4\pi i V_3\int dzd\bar{z}\left(\langle\bar{D}\bar{Z},\left[Z,X^I,X^J\right]\rangle-2\langle\bar{D}X^I,\left[Z,\bar{Z},X^J\right]\rangle\right)\\
\mathcal{Z}^{IJ}_{z} =&-4\pi i V_3\int dzd\bar{z}\left(\langle DZ,\left[\bar{Z},X^I,X^J\right]\rangle+2\langle DX^I,\left[Z,\bar{Z},X^J\right]\rangle\right)\\
\mathcal{Z}^{IJK}_{\bar{Z}} =&\; 6i\pi V_3\int dzd\bar{z}\langle\left[Z,X^I,X^J\right],\left[Z,\bar{Z},X^K\right]\rangle\\
\mathcal{Z}^{IJK}_{Z} =&\; 6i\pi V_3\int dzd\bar{z}\langle\left[\bar{Z},X^I,X^J\right],\left[Z,\bar{Z},X^K\right]\rangle\\
\mathcal{Z}^{IJ}_{z\bar{z}} =&-32\sqrt{2}\pi V_3\int dzd\bar{z}\langle DX^I,\bar{D}X^J\rangle\\
\mathcal{Z}^{IJ}_{Z\bar{Z}} =&\; 4\sqrt{2}\pi V_3\int dzd\bar{z}\left(2\langle\left[Z,X^I,X^K\right],\left[\bar{Z},X^J,X^K\right]\rangle\right.\\
&\qquad\qquad\left.+i\langle\left[Z,X^I,X^J\right],D_+\bar{Z}\rangle+i\langle\left[\bar{Z},X^I,X^J\right],D_+Z\rangle\right)\\
\mathcal{Z}^{IJK}_{\bar{Z}z} =&\; 24\sqrt{2}\pi V_3\int dzd\bar{z}\langle\left[Z,X^I,X^J\right],DX^K\rangle\\
\mathcal{Z}^{IJK}_{\bar{Z}\bar{z}} =&\; 24\sqrt{2}\pi V_3\int dzd\bar{z}\langle\left[Z,X^I,X^J\right],\bar{D}X^K\rangle\\
\mathcal{Z}^{IJK}_{Zz} =&\; 24\sqrt{2}\pi V_3\int dzd\bar{z}\langle\left[\bar{Z},X^I,X^J\right],DX^K\rangle\\
\mathcal{Z}^{IJK}_{Z\bar{z}} =&\; 24\sqrt{2}\pi V_3\int dzd\bar{z}\langle\left[\bar{Z},X^I,X^J\right],\bar{D}X^K\rangle\\
\mathcal{Z}^{IJKL} =&-12\sqrt{2}\pi V_3\int dzd\bar{z}\langle\left[Z,X^I,X^J\right],\left[\bar{Z},X^K,X^L\right]\rangle\ ,
\end{aligned}
\tag{3.2.26}
$$

where anti-symmetrization on all free I,J,K,L indices is understood.

3.3 Reduction to Dynamics on Moduli Space

I now turn to an analysis of the dynamical equations that were found above. I view x^+ as 'time' and take the Hamiltonian to be $-\mathcal{P}_+$. The equations found are a novel system of differential equations for a set of 3-algebra valued fields (X^I,Z,H,Ψ_+,Ψ_-) along with a Lie-algebra valued gauge field $(A_+,A_z,A_{\bar{z}})$ all of which depend on two space and one null direction $(z,\bar{z},x^+)$ and are invariant under 16 supersymmetries generated by $\mathcal{Q}_+$ and $\mathcal{Q}_-$.

3.3.1 Abelian Case

To gain some insight it is helpful to first solve the abelian case where the triple product vanishes and the gauge fields are set to zero. The equations of motion are simply

$$\begin{aligned}\partial_+\Psi_+ + \sqrt{2}\hat{\Gamma}_z\bar{\partial}\Psi_- + \sqrt{2}\hat{\Gamma}_{\bar{z}}\partial\Psi_- &= 0\\ \sqrt{2}\hat{\Gamma}_z\bar{\partial}\Psi_+ + \sqrt{2}\hat{\Gamma}_{\bar{z}}\partial\Psi_+ &= 0\\ \bar{\partial}Z &= 0\\ \bar{\partial}\partial X^I &= 0\\ \partial_+^2\bar{Z} + 4\bar{\partial}H &= 0\ .\end{aligned} \tag{3.3.1}$$

The solutions to these equations are readily seen to be given by taking Z to be an arbitrary x^+ dependent holomorphic function of z, and X^I can be taken to be the real part of an arbitrary x^+ dependent holomorphic function. For H we find

$$H = h - \frac{1}{4}\int_0^{\bar{z}} \partial_+^2\bar{Z}(\bar{z}')d\bar{z}'\ . \tag{3.3.2}$$

where h is a holomorphic function which also has an arbitrary dependence on x^+. Looking at the fermions one finds

$$\begin{aligned}\Psi_+ &= \eta_+ + \bar{\eta}_+\\ \Psi_- &= \eta_- + \bar{\eta}_- - \frac{1}{\sqrt{2}}\int_0^z \hat{\Gamma}_z\partial_+\eta_+(z')dz' - \frac{1}{\sqrt{2}}\int_0^{\bar{z}} \hat{\Gamma}_{\bar{z}}\partial_+\bar{\eta}_+(\bar{z}')d\bar{z}'\ ,\end{aligned} \tag{3.3.3}$$

where $\eta_\pm$ are spinors which satisfy

$$\hat{\Gamma}_{\bar{z}}\eta_\pm = 0\ . \tag{3.3.4}$$

and which are also holomorphic functions with some x^+ dependence.

Thus the solution space is a set of holomorphic functions with arbitrary x^+dependence. To recover some physics I note that for generic solutions the energy P_+ will diverge due to the poles in the holomorphic functions. Thus on physical grounds one should take all holomorphic functions to be constant. In this case P_+ will still diverge due to the integral over z however one could imagine putting the theory on a torus and reducing the system to a quantum mechanical model. In that case global consistency requires that

$$\partial_+\Psi_+ = 0 \qquad \partial_+^2 Z = 0 \ . \tag{3.3.5}$$

In this way one sees the recovery of the familiar free-dynamics of Ψ_+ and Z, although the x^+ dependence of X^I, H and Ψ_- remain unconstrained. Looking at the on-shell supersymmetry in this case one sees that

$$\begin{aligned}
\delta\Psi_+ &= -i(\hat{\Gamma}_Z\partial_+ Z - \bar{\hat{\Gamma}}_{\bar{Z}}\partial_+\bar{Z})\epsilon_+ \\
\delta\Psi_- &= \frac{i}{l^3}\left(\hat{\Gamma}_Z\partial_+ Z - \hat{\Gamma}_{\bar{Z}}\partial_+\bar{Z}\right)\epsilon_- + 2\sqrt{2}i\left(\hat{\Gamma}_{\bar{z}}\hat{\Gamma}_{\bar{Z}}H - \hat{\Gamma}_z\hat{\Gamma}_Z\bar{H}\right)\epsilon_+ \\
\delta Z &= 2\sqrt{2}\epsilon_+^T\hat{\Gamma}_{\bar{Z}}\Psi_+ \\
\delta X^I &= i\epsilon_+^T\hat{\Gamma}^I\Psi_- + i\epsilon_-^T\hat{\Gamma}^I\Psi_+ \\
\delta H &= \epsilon_+^T\hat{\Gamma}_z\hat{\Gamma}_Z\partial_+\Psi_- \ .
\end{aligned} \tag{3.3.6}$$

Thus under $\mathcal{Q}_+$ (Z,Ψ_+) and (X^I,H,Ψ_-) form separate multiplets whereas under $\mathcal{Q}_-$ (Z,Ψ_+) and H are invariant but (X^I,Ψ_-) transform into in (Ψ_+,Z).

Even in the non-abelian case one sees that there are no standard kinetic terms for X^I, H and Ψ_-. Indeed there are no D_+ derivatives on H or Ψ_- and D_+ only appears linearly on X^I and within a triple product. Thus I will interpret X^I, H and Ψ_- as, possibly x^+-dependent, background fields. Given a particular choice of these fields as functions of z and x^+ the equations of motion then determine the behaviour of Z and Ψ_+.

3.3.2 Vacua of the Non-Abelian System

Next I consider the form of the supersymmetry algebra. Here one sees that $\mathcal{Q}_-$ is broken unless

$$\mathcal{W} = 0 \ . \tag{3.3.7}$$

However this implies that $DZ = 0$ and hence $F_{z\bar{z}}(Z) = 0$. This effectively reduces the system back to the abelian case. Thus in what follows I assume that $\mathcal{Q}_-$ is broken and set $\epsilon_- = 0$. Then examine the system where only $\mathcal{Q}_+$ acts dynamically. The role of $\mathcal{Q}_-$ can then be thought of as mapping between different backgrounds defined by choices of X^I, H and Ψ_-.

In this work I will only consider backgrounds which preserve all of the $\mathcal{Q}_+$ supersymmetries. In particular for a generic ϵ_+ one sees that such backgrounds are of the form $\Psi_- = 0$, $H = 0$ with $D_+X^I = 0$ and $[X^I, X^J, X^K] = 0$. Henceforth I will only consider such solutions. In this case the gauge fields are also invariant under $\mathcal{Q}_+$. Therefore the dynamical fields are Z and Ψ_+. For simplicity I will also set $\Psi_+ = 0$ with the understanding that their dynamics can be recovered by applying the $\mathcal{Q}_+$ supersymmetry to the bosonic equations.

To begin I note that the ground states with $\mathcal{P}_+ = 0$ correspond to

$$DX^I = 0 \qquad [Z, X^I, X^J] = 0 \qquad D_+Z = 0 \ , \tag{3.3.8}$$

and such states are indeed invariant under $\mathcal{Q}_+$ and can have a non-vanishing $\mathcal{W}$. The equations of motion reduce to simply

$$\begin{aligned} \bar{D}Z &= 0 \\ F_{z\bar{z}}(\cdot) &= -\frac{1}{4}\left[X^I, \left[Z, \bar{Z}, X^I\right], \cdot\right] \ . \end{aligned} \tag{3.3.9}$$

Since the X^I are covariantly constant: $DX^I = \bar{D}X^I = 0$ this equation is essentially just that of a Hitchin system [72] but in a three-algebra format as I now detail.

To continue I consider the specific case of a positive-definite 3-algebra with generators T^A, $A = 1, 2, 3, 4$ whose inner-product is $\langle T^A, T^B\rangle = \delta^{AB}$ and triple product

$$[T^A, T^B, T^C] = \frac{2\pi}{k}\varepsilon^{ABCD}T^D \ , \tag{3.3.10}$$

where k is a constant (usually taken to be integer). The gauge field takes values in $so(4) = su(2) \oplus su(2)$ and the fields X^I and Z are in the vector of $SO(4)$. Solutions for X^I that satisfy $[X^I, X^J, X^K] = 0$ can be expanded in terms of two constant

$SO(6)$ vectors u^I, v^I:

$$X^I = u^I T^3 + v^I T^4 \ . \tag{3.3.11}$$

For generic choices of u^I and v^I the gauge group is completely broken and the vacuum equations have no non-trivial solutions. In particular Z is also restricted to lie in the T^3 and T^4 directions of the 3-algebra and the gauge field is locally flat. As with the abelian case above all the non-zero components of the fields are given by holomorphic functions. However demanding that $\mathcal{W}$ and $\mathcal{P}_+$ be finite requires that these holomorphic functions are constant and space is compactified.

However if one takes all the X^I to be aligned in the 3-algebra, say $X^I = v^I T^4$ then there is an unbroken $SO(3)$. If one expands $Z = \sum Z_A T^A$ then $DX^I = \bar{D}X^I = 0$ implies $\partial v^I = \bar{\partial} v^I = 0$ and $A_{z4}{}^b = A_{za}{}^4 = 0$, $a, b, = 1, 2, 3$. The solutions are then given by

$$\begin{aligned} \bar{\mathbf{D}}\mathbf{Z} &= 0 \\ \mathbf{F}_{z\bar{z}} &= -\frac{\pi^2 |v|^2}{k^2}[\mathbf{Z}, \bar{\mathbf{Z}}] \ , \end{aligned} \tag{3.3.12}$$

where a bold face indicates that the components are orthogonal to T^4 in the three-algebra and re-expressed as elements of the $SO(3)$ Lie-algebra: $(\mathbf{Z})^a{}_b = \varepsilon^{ca}{}_b Z_c$, $\mathbf{D}\mathbf{Z} = \partial \mathbf{Z} - [\mathbf{A}, \mathbf{Z}]$. Furthermore [,] is the usual Lie-bracket.[2] In other words bold-faced fields can be viewed as taking values in the unbroken $su(2)$ Lie-algebra. This is precisely the Hitchin system for gauge algebra $su(2)$ [72]. The equations of motion allow for Z_4 to be any holomorphic function but demanding that $\mathcal{W}$ is finite implies that Z_4 is constant. Thus the vacuum solutions are in a one-to-one correspondence with solutions to the Hitchin system for $su(2)$.

It is useful to recall here that the Hitchin system itself is the dimensional reduction of the the four-dimensional self-duality equations to two-dimensions. In particular, define

$$\mathbf{A}_3 = \frac{2\pi|v|}{k}\frac{\mathbf{Z} - \bar{\mathbf{Z}}}{2i} \qquad \mathbf{A}_4 = \frac{2\pi|v|}{k}\frac{\mathbf{Z} + \bar{\mathbf{Z}}}{2} \ . \tag{3.3.13}$$

[2]Note that the conventions for matrix multiplication are somewhat unusual here: $(MN)^A{}_B X_A = M^C{}_B N^A{}_C X_A$

Equation (3.3.12) can then be written as (recall that $z = x^1 + ix^2$)

$$\begin{aligned}\mathbf{F}_{13} &= -\mathbf{F}_{24}\\ \mathbf{F}_{23} &= \mathbf{F}_{14}\\ \mathbf{F}_{12} &= \mathbf{F}_{34}\ ,\end{aligned} \tag{3.3.14}$$

which are indeed the self-duality conditions and $\mathcal{W}$ is the dimensional reduction of instanton number and as such is no longer integer.

3.3.3 Dynamical Evolution

Next, allow for x^+ dependence and allow Z to be dynamical, although I continue to restrict to the $\mathcal{Q}_+$ invariant sector: $D_+X^I = H = [X^I, X^J, X^K] = \Psi_- = 0$. For simplicity, also set $\Psi_+ = 0$ with the understanding its dynamics can be restored using the $\mathcal{Q}_+$ supersymmetry. Keeping $X^I = v^I T^4$ and $Z_4 = w$ this requires that $\partial_+ v^I = 0$ and $A^a_{+4} = -A^4_{+a} = 0$. It is helpful then to rewrite the equations for the various remaining fields which I now express in their $su(2)$-valued form.

To begin observe that $(\mathbf{B})^b{}_c = \varepsilon^{ab}{}_c A^a_{z4}$ is not necessarily zero since DX^I need not vanish. This implies that the holomorphic constraint $\bar{D}Z = 0$ leads to the equations

$$\begin{aligned}\bar{\partial} w + \frac{1}{2}\mathrm{tr}(\bar{\mathbf{B}}\mathbf{Z}) &= 0\\ \bar{\mathbf{D}}\mathbf{Z} + \bar{\mathbf{B}} w &= 0\ ,\end{aligned} \tag{3.3.15}$$

for the $A = 4$ and $A = a$ components respectively. Thus a non-zero w and $\mathbf{B}$ lead to change in the holomorphic constraint on $\mathbf{Z}$.

Next I recall that the Hitchin equation (3.3.12) which arose from the $(C, D) = (c, d)$ component of the $F_{z\bar{z}}$ equation now becomes

$$\mathbf{F}_{z\bar{z}} = -\frac{\pi^2|v|^2}{k^2}[\mathbf{Z}, \bar{\mathbf{Z}}] + [\mathbf{B}, \bar{\mathbf{B}}] + \frac{i}{4}\left(\frac{2\pi}{k}\right)(w\mathbf{D}_+\bar{\mathbf{Z}} + \bar{w}\mathbf{D}_+\mathbf{Z} - \bar{\mathbf{Z}}\partial_+ w - \mathbf{Z}\partial_+\bar{w})\ , \tag{3.3.16}$$

where

$$\mathbf{F}_{z\bar{z}} = \partial\bar{\mathbf{A}} - \bar{\partial}\mathbf{A} - [\mathbf{A}, \bar{\mathbf{A}}]\ . \tag{3.3.17}$$

If one examines the $(C,D) = (c,4)$ component of the $F_{z\bar{z}}$ equation one finds

$$\mathbf{D}\bar{\mathbf{B}} - \bar{\mathbf{D}}\mathbf{B} = -\frac{i}{4}\left(\frac{2\pi}{k}\right)\left([\mathbf{Z}, \mathbf{D}_+\bar{\mathbf{Z}}] + [\bar{\mathbf{Z}}, \mathbf{D}_+\mathbf{Z}]\right) \ . \tag{3.3.18}$$

From the F_{+z} equation one learns that

$$\begin{aligned} \mathbf{D}_+\mathbf{B} &= 0 \\ \partial_+\mathbf{A} - \mathbf{D}\mathbf{A}_+ &= -\frac{2\pi i}{k}|v|^2\mathbf{B} \ , \end{aligned} \tag{3.3.19}$$

due to the $(C,D) = (c,d)$ and $(C,D) = (c,4)$ components respectively. From the $(D\bar{D} + \bar{D}D)X^I$ equation I find

$$\begin{aligned} \partial\bar{\partial}v^I + \frac{1}{2}\text{tr}(\bar{\mathbf{B}}\mathbf{B})v^I &= 0 \\ (\mathbf{D}\bar{\mathbf{B}} + \bar{\mathbf{D}}\mathbf{B})v^I + 2\mathbf{B}\bar{\partial}v^I + 2\bar{\mathbf{B}}\partial v^I &= \frac{i}{4}\left(\frac{2\pi}{k}\right)\left([\mathbf{Z}, \mathbf{D}_+\bar{\mathbf{Z}}] - [\bar{\mathbf{Z}}, \mathbf{D}_+\mathbf{Z}]\right)v^I \ , \end{aligned} \tag{3.3.20}$$

arising from the $A = 4$ and $A = a$ components respectively. Lastly I also simply find

$$\mathbf{D}_+^2\mathbf{Z} = 0 \qquad \partial_+^2 w = 0 \ . \tag{3.3.21}$$

One sees that non-vanishing $\mathbf{B}$ and w lead to a z-dependent v^I and hence to a modification of Hitchin's system.

Our approach here is to treat X^I and hence v^I as a background field. Elementary manipulations of the first equation in (3.3.20) shows that

$$\oint v^I dv^I = \int \frac{1}{2}\text{tr}(\mathbf{B}^\dagger\mathbf{B})|v|^2 + \int |\partial v^I|^2 + |\bar{\partial}v^I|^2 \geq 0 \ . \tag{3.3.22}$$

Thus if one is interested in solutions for which v^I approaches a non-zero constant value at infinity plus subleading terms then the left hand side vanishes. Therefore $\mathbf{B} = 0$ and v^I is constant. I first consider the case when $w = 0$. In this case one sees that Hitchin's equation is preserved for all time. Thus any dynamical motion can only take place on the moduli space of solutions to Hitchin's system. In addition the remaining dynamical equations are

$$[\mathbf{Z}, \mathbf{D}_+\bar{\mathbf{Z}}] = 0, \qquad \partial_+\mathbf{A} = \mathbf{D}\mathbf{A}_+ \ , \qquad \mathbf{D}_+^2\mathbf{Z} = 0 \ . \tag{3.3.23}$$

To understand these equations I recall that $(\mathbf{A}, \mathbf{Z})$ are required to solve the Hitchin equations for all x^+. Thus motion can only take place on the moduli space of solutions so that under $x^+ \to x^+ + \epsilon$,

$$\delta\mathbf{A} = \partial_+\mathbf{A}\epsilon \qquad \delta\mathbf{Z} = \partial_+\mathbf{Z}\epsilon \ , \tag{3.3.24}$$

where $\delta\mathbf{A}$ and $\delta\mathbf{Z}$ are fluctuations of the solution to Hitchin's equations: *i.e.* solutions to the linearised Hitchin equations. In particular these linearised equations are

$$\begin{aligned} \mathbf{D}\partial_+\bar{\mathbf{A}} - \bar{\mathbf{D}}\partial_+\mathbf{A} &= -\frac{\pi^2}{k^2}|v|^2\left([\partial_+\mathbf{Z}, \bar{\mathbf{Z}}] + [\mathbf{Z}, \partial_+\bar{\mathbf{Z}}]\right) \\ \bar{\mathbf{D}}\partial_+\mathbf{Z} - [\partial_+\bar{\mathbf{A}}, \mathbf{Z}] &= 0 \ . \end{aligned} \tag{3.3.25}$$

Using the second equation in (3.3.23) one sees that

$$\begin{aligned} \mathbf{D}\partial_+\bar{\mathbf{A}} - \bar{\mathbf{D}}\partial_+\mathbf{A} &= (\mathbf{D}\bar{\mathbf{D}} - \bar{\mathbf{D}}\mathbf{D})\mathbf{A}_+ \\ &= -[\mathbf{F}_{z\bar{z}}, \mathbf{A}_+] \\ &= \frac{\pi^2}{k^2}|v|^2[[\mathbf{Z}, \bar{\mathbf{Z}}], \mathbf{A}_+] \\ &= -\frac{\pi^2}{k^2}|v|^2([[\mathbf{A}_+, \mathbf{Z}], \bar{\mathbf{Z}}] + [\mathbf{Z}, [\mathbf{A}_+, \bar{\mathbf{Z}}]]) \\ &= -\frac{\pi^2}{k^2}|v|^2\left([\partial_+\mathbf{Z}, \bar{\mathbf{Z}}] + [\mathbf{Z}, \partial_+\bar{\mathbf{Z}}]\right) \ , \end{aligned} \tag{3.3.26}$$

where in the last line I used the first equation in (3.3.23). Thus (3.3.23) imply the first equation in (3.3.25). Using (3.3.23) the second equation in (3.3.25) becomes simply

$$\bar{\mathbf{D}}\mathbf{D}_+\mathbf{Z} = 0 \ . \tag{3.3.27}$$

Thus the dynamical equations (3.3.23) along with (3.3.27) describe motion on the Hitchin moduli space.

To continue I note that we do not want to consider motion that arises from gauge transformations: $\delta\mathbf{A} = \mathbf{D}\omega$, $\delta\mathbf{Z} = [\omega, \mathbf{Z}]$. Therefore I impose that the fluctuations are orthogonal to gauge transformations[3]:

$$-\frac{1}{2}\mathrm{tr}\int dzd\bar{z}\ \left[2\bar{\mathbf{D}}\omega\delta\mathbf{A}+2\mathbf{D}\omega\delta\bar{\mathbf{A}}+\frac{2\pi^2}{k^2}|v|^2\left([\omega,\bar{\mathbf{Z}}]\delta\mathbf{Z}+[\omega,\mathbf{Z}]\delta\bar{\mathbf{Z}}]\right)\right]=0\ . \quad (3.3.28)$$

Integrating by parts and demanding that ω is arbitrary gives the condition

$$\mathbf{D}\delta\bar{\mathbf{A}}+\bar{\mathbf{D}}\delta\mathbf{A}=\frac{\pi^2}{k^2}|v|^2\left([\mathbf{Z},\delta\bar{\mathbf{Z}}]+[\bar{\mathbf{Z}},\delta\mathbf{Z}]\right)\ . \quad (3.3.29)$$

Identifying $\delta\mathbf{A}=\partial_+\mathbf{A}\epsilon$, $\delta\mathbf{Z}=\partial_+\mathbf{Z}\epsilon$ and combining with the first equation in (3.3.25) gives the gauge fixing condition:

$$\bar{\mathbf{D}}\partial_+\mathbf{A}=\frac{\pi^2}{k^2}|v|^2[\mathbf{Z},\partial_+\bar{\mathbf{Z}}]\ , \quad (3.3.30)$$

or equivalently using (3.3.23)

$$\bar{\mathbf{D}}\mathbf{D}\mathbf{A}_+=\frac{\pi^2}{k^2}|v|^2[\mathbf{Z},[\mathbf{A}_+,\bar{\mathbf{Z}}]]\ . \quad (3.3.31)$$

Thus for the background $X^I=v^IT^4$, $Z_4=0$ the whole dynamical system is reduced to motion on the moduli space of solutions to Hitchin's equations with the dynamical equations (3.3.23), (3.3.27) and gauge fixing condition (3.3.31). The Hamiltonian is given by $\mathcal{H}=-\mathcal{P}_+$ which in turn is simply that of a σ-model on the

[3]This is just the reduction of the standard instanton moduli space gauge fixing condition $\mathrm{tr}\int A_1\delta A_1+...+A_4\delta A_4$ for the four-dimensional gauge field defined in (3.3.13).

moduli space:

$$
\begin{aligned}
\mathcal{H} &= \pi \int dz d\bar{z} \langle D_+ Z, D_+ \bar{Z} \rangle \\
&= -\frac{\pi}{2} \mathrm{tr} \int dz d\bar{z} \left((\partial_+ \mathbf{Z} - [\mathbf{A}_+, \mathbf{Z}])(\partial_+ \bar{\mathbf{Z}} - [\mathbf{A}_+, \bar{\mathbf{Z}}]) \right) \\
&= -\frac{\pi}{2} \mathrm{tr} \int dz d\bar{z} \left(\partial_+ \mathbf{Z} \partial_+ \bar{\mathbf{Z}} - \mathbf{A}_+ [\bar{\mathbf{Z}}, \partial_+ \mathbf{Z}] - \mathbf{A}_+ [\mathbf{Z}, \partial_+ \bar{\mathbf{Z}}] + \mathbf{A}_+ [\mathbf{Z}, [\mathbf{A}_+, \bar{\mathbf{Z}}]] \right) \\
&= -\frac{\pi}{2} \mathrm{tr} \int dz d\bar{z} \left(\partial_+ \mathbf{Z} \partial_+ \bar{\mathbf{Z}} - \frac{1}{2} \mathbf{A}_+ [\bar{\mathbf{Z}}, [\mathbf{A}_+, \mathbf{Z}]] - \frac{1}{2} \mathbf{A}_+ [\mathbf{Z}, [\mathbf{A}_+, \bar{\mathbf{Z}}]] \right) \\
&= -\frac{\pi}{2} \int dz d\bar{z} \left(\partial_+ \mathbf{Z} \partial_+ \bar{\mathbf{Z}} - \frac{k^2}{2\pi^2 |v|^2} \mathbf{A}_+ \mathbf{D} \bar{\mathbf{D}} \mathbf{A}_+ - \frac{k^2}{2\pi^2 |v|^2} \mathbf{A}_+ \bar{\mathbf{D}} \mathbf{D} \mathbf{A}_+ \right) \\
&= -\frac{k^2}{2\pi |v|^2} \mathrm{tr} \int dz d\bar{z} \left(\frac{\pi^2 |v|^2}{k^2} \partial_+ \mathbf{Z} \partial_+ \bar{\mathbf{Z}} + \partial_+ \mathbf{A} \partial_+ \bar{\mathbf{A}} \right) \\
&= \frac{k^2}{2\pi |v|^2} g_{mn} \partial_+ \xi^m \partial_+ \xi^n \ ,
\end{aligned}
\tag{3.3.32}
$$

where I have used the relations $[\mathbf{Z}, \partial_+ \bar{\mathbf{Z}}] = [\mathbf{Z}, [\mathbf{A}_+, \bar{\mathbf{Z}}]]$, $\bar{\mathbf{D}} \partial_+ \mathbf{A} = \frac{\pi^2}{k^2} |v|^2 [\mathbf{Z}, \partial_+ \bar{\mathbf{Z}}]$ and $\partial_+ \mathbf{A} = \mathbf{D} \mathbf{A}_+$. Furthermore ξ^m are the moduli space coordinates and

$$
g_{mn} = -\frac{1}{2} \mathrm{tr} \int dz d\bar{z} \left(\delta_m \mathbf{A}_1 \delta_n \mathbf{A}_1 + \delta_m \mathbf{A}_2 \delta_n \mathbf{A}_2 + \delta_m \mathbf{A}_3 \delta_n \mathbf{A}_3 + \delta_m \mathbf{A}_4 \delta_n \mathbf{A}_4 \right) \ ,
\tag{3.3.33}
$$

is the natural metric on the moduli space. As shown by Hitchin [72] this space is hyper-Kahler and therefore, by standard arguments, the dynamics can be extended to include fermions in such a way as to preserve the 8 supersymmetries generated by $\mathcal{Q}_+$.

Next I can consider the effect of a non-zero w but still keeping v^I constant and hence $\mathbf{B} = 0$. One sees that for static solutions with $\partial_+ = \mathbf{A}_+ = 0$ one still reduces to Hitchin's system however for $\mathbf{A}_+, \partial_+ \neq 0$ there is a modifcation. To see what happens one can differentiate (3.3.16) with respect to ∂_+ to find (recall that $\mathbf{D}_+^2\mathbf{Z} = \partial_+^2 w = 0$):

$$\begin{aligned}
\mathbf{D}\partial_+\bar{\mathbf{A}} - \bar{\mathbf{D}}\partial_+\mathbf{A} &= -\frac{\pi^2|v|^2}{k^2}\left([\partial_+\mathbf{Z}, \bar{\mathbf{Z}}] + [\mathbf{Z}, \partial_+\bar{\mathbf{Z}}]\right) \\
&\quad + \frac{i}{4}\left(\frac{2\pi}{k}\right)\left(w\partial_+\mathbf{D}_+\bar{\mathbf{Z}} + \bar{w}\partial_+\mathbf{D}_+\mathbf{Z} - \partial_+ w[\mathbf{A}_+, \bar{\mathbf{Z}}] - \partial_+\bar{w}[\mathbf{A}_+, \mathbf{Z}]\right) \\
&= \frac{\pi^2}{k^2}|v|^2[[\mathbf{Z}, \bar{\mathbf{Z}}], \mathbf{A}_+] \\
&\quad - \frac{i}{4}\left(\frac{2\pi}{k}\right)\left[w\mathbf{D}_+\bar{\mathbf{Z}} + \bar{w}\mathbf{D}_+\mathbf{Z} - \partial_+ w\bar{\mathbf{Z}} - \partial_+\bar{w}\mathbf{Z}, \mathbf{A}_+\right] . \qquad (3.3.34)
\end{aligned}$$

This generalises the first equation in (3.3.25) and the rest of the analysis continues as before. One sees that the analysis in (3.3.26) still goes through and one still finds that (3.3.23) imply the first equation in (3.3.25). However (3.3.31) is now modified to

$$\begin{aligned}
\bar{\mathbf{D}}\mathbf{D}\mathbf{A}_+ &= \frac{\pi^2}{k^2}|v|^2[\mathbf{Z}, [\mathbf{A}_+, \bar{\mathbf{Z}}]] \\
&\quad - \frac{i}{8}\left(\frac{2\pi}{k}\right)\left[w\mathbf{D}_+\bar{\mathbf{Z}} + \bar{w}\mathbf{D}_+\mathbf{Z} - w\bar{\mathbf{Z}} - \bar{w}\mathbf{Z}, \mathbf{A}_+\right] . \qquad (3.3.35)
\end{aligned}$$

The rest of the equations remain unchanged. In particular the Hamiltonian is the same except for an additional term in $\mathcal{P}_+$:

$$\pi\int D_+Z_4 D_+\bar{Z}_4 = \pi\int dz d\bar{z}\partial_+ w\partial_+\bar{w} \ . \qquad (3.3.36)$$

This will diverge unless $\partial_+ w = 0$ as w is holomorphic (although it would be finite for constant w if placed on a compact Riemann surface).

Lastly one can quantize the system in a natural way by considering wavefunctions $\psi(\xi^m)$ and replacing

$$\partial_+\xi^m \to -i\frac{\partial\psi}{\partial\xi^m} \ . \qquad (3.3.37)$$

Thus the dynamics reduces to quantum mechanics on the Hitchin moduli space.

3.4 Physical Interpretation

So far in this chapter I have solved the constraints of the $(2,0)$ superalgebra of [147] for a particular choice of three-form $C = l^3 dx^3 \wedge dx^4 \wedge dx^+$. I showed that the resulting system of equations had vacuum configurations consisting of solutions to the Hitchin system on $\mathbf{R}^2$. I also showed that the dynamical evolution consisted of motion on the moduli space $\mathcal{H}_K(su(2), \mathbb{R}^2)$ of such solutions. Here $\mathcal{H}_n(\mathfrak{g}, \Sigma)$ denotes the moduli space of the charge n Hitchin system with gauge algebra $\mathfrak{g}$ on a Riemann surface Σ. Therefore it is of interest to see how this construction fits in with other known descriptions of M-branes.

To begin with recall that to solve the constraints of the original $(2,0)$ algebra it was necessary to dimensionally reduce the full six-dimensional system on x^3, x^4 and x^-. However it is clear from the subsequent analysis that the resulting system still carries information about the momentum around x^- in the form of the topological term $\mathcal{W} \sim \int T_{--}$. Thus one should view the system as two M5-branes compactified on $\mathbb{T}^2 \times S^1_-$ but with a fixed null momentum $\mathcal{P}_- \sim \mathcal{W}$.

One can view a null compactification as a limit of a boosted spacelike compactification where x^5 is taken to be compact with a radius that vanishes so that in the limit of a null boost the radius R_- remains finite. Therefore I review the case where $C = l^3 dx^3 \wedge dx^4 \wedge dx^5$ is spacelike and the constraints imply that the fields have no dependence on x^3, x^4, x^5. It was shown in [147] that the $(2,0)$ superalgebra reduces to the description of two M2-branes with a transverse $\mathbb{R}^8$. From a brane perspective one can think of this as a toroidal compactification on x^3, x^4, x^5, sending all the radii to zero, accompanied by a U-duality transformation which decompactifies the dual torus. This can be thought of as an M-theory version of T-duality that takes N M5-branes wrapped on $\mathbb{T}^3$ to N M2-branes which are transverse to a dual $\hat{\mathbb{T}}^3$.[4] In particular the U-duality required consists of reducing to string theory on x^5, leading to N D4-branes wrapped on a $\mathbb{T}^2$ with a coupling $g^2_{YM} \sim R_5$, and then performing T-dualities along x^3 and x^4 to find N D2-branes with a transverse $\hat{\mathbb{T}}^2 \times \mathbb{R}^5$ where the radii are $\hat{R}_3 = \alpha'/R_3$ and $\hat{R}_4 = \alpha'/R_4$ and the coupling constant is $\hat{g}^2_{YM} \sim R_5/R_3R_4$. If one now shrinks the original radii to zero one obtains the strong coupling limit of N $D2$-branes in a transverse $\mathbb{R}^7$ or equivalently N M2-branes in a transverse $\mathbb{R}^8$.

[4] For the sake of generality here I have considered an arbitrary number of M-branes whereas the results found above only concern the case of $N = 2$.

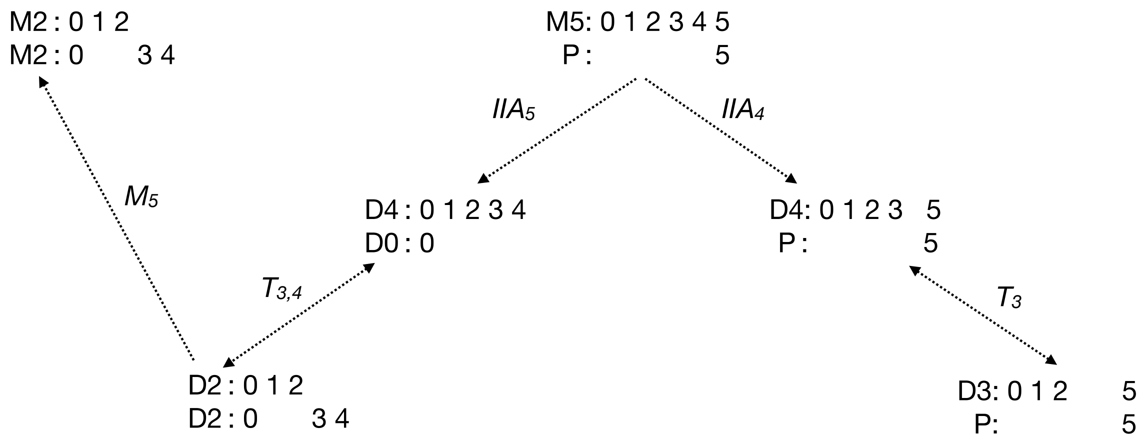

Figure 3.1: U-dualities of an M5 with momentum. IIA_n indicates reduction to string theory along x^n, T_n T-duality along x^n and M_n lift to M-theory along x^n.

Now repeat these steps with K units of momentum along x^5. In addition to the N D4-branes one also finds the momentum modes becoming K D0-branes as described in Section 1.2. After T-duality these become K D2-branes along x^3, x^4. Taking all the radii to zero leads to N M2-branes along x^0, x^1, x^2 and K M2-branes along x^0, x^3, x^4. The Hitchin system can then be thought of as the BPS condition for K M2-branes intersecting the original N M2-branes, generalising the familiar abelian holomorphic condition $\bar{\partial}Z = 0$ for intersecting branes. One also sees that there will be an $SO_L(2) \times SO_R(2) \times SO_R(6)$ symmetry from rotations in the (x^1, x^2), (x^3, x^4) and $(x^5, ..., x^{10})$ planes respectively.

Lastly one needs to perform the light-like boost along x^5 which is transverse to all the M2-branes. In terms of static gauge this corresponds to replacing X^5 with $-vx^0+X^5$ and taking the limit $v \to 1$. For $v \neq 0$ this will break the $SO_R(6)$ symmetry of the total transverse space to $SO(5)$. However one can see that the breaking only occurs through the time derivative kinetic terms. The spatial gradient terms will remain invariant under $SO_R(6)$. The interaction terms also remain invariant since $X^5 \to -vx^0 + X^5$ is a shift by the centre of mass degree of freedom which is non-interacting[5]. If one takes the limit $v \to 1$ then the M2-brane tension vanishes, the kinetic terms diverge and we are forced to set them to zero. Thus the $SO_R(6)$ symmetry is restored. In addition one can allow the moduli to evolve such that $\partial_0\xi^m \sim \mathcal{O}(\sqrt{1-v^2})$. In this case the $SO_R(6)$ symmetry remains unbroken as these moduli are invariant under rotations of the total transverse space. In the limit that $v \to 1$ the Manton approximation of slow motion on the moduli space of solutions becomes exact and the dynamics reduces exactly to motion on $\mathcal{H}_K(su(2), \mathbb{R}^2)$.

This agrees with the results found in the previous section. Stated somewhat differently boosting the intersecting M2-branes leads to 'fast' modes corresponding to the over-all transverse scalars X^I (what were called the background fields before) and 'slow' modes corresponding to the moduli ξ^m. Time evolution of the 'fast' modes breaks $SO_R(6)$ to $SO(5)$ but time evolution of the 'slow' modes does not. Thus the $(2, 0)$ system obtained above can be viewed as describing the 'slow' modes, with the 'fast' modes frozen or integrated out (*i.e.* set to their expectation values).

I now comment on a separate but related description of N M5-branes on $\mathbb{T}^2 \times S^1_-$. In particular if one first compactifies on $\mathbb{T}^2$. As is well known, reduction of the

[5]This is clear for D2-branes where the centre of mass degree of freedom is given by the identity matrix and all interactions are through commutators. This degree of freedom can be somewhat subtle in interacting M2-brane models but ultimately one expects this statement to remain true.

A_{N-1} $(2,0)$ theory on a torus of vanishing area (but fixed shape) leads to maximally supersymmetric $U(N)$ Yang-Mills. More precisely one can reduce to string theory on x^4 to obtain N D4-branes with coupling $g^2_{YM} \sim R_4$ and then T-dualise along x^3 to find N D3-branes with finite coupling $g^2 \sim R_4/R_3$. Lastly, introduce K units of null momentum along x^5 which leaves a manifest $SO(2) \times SO(6)$ symmetry that arises from rotations in the (x^1, x^2) and $(x^3, x^6, x^7, x^8, x^9, x^{10})$ planes respectively. This is the set-up for a DLCQ construction of four-dimensional maximally supersymmetric Yang-Mills. This was given in [153] in terms of the quantum mechanics on $\mathcal{H}_N(u(K), \hat{\mathbb{T}}^2)$ where $\hat{\mathbb{T}}^2$ is an auxiliary two-torus. Various details of this system have been studied in detail more recently in [154] and see also [152] for an alternative description.

These two descriptions differ by a T-duality along x^4 as well as a U-duality corresponding to the choice of M-theory direction (a '$9-11$ flip' that swaps x^4 with x^5). However it is also possible that the two descriptions involve different choices of 'fast' and 'slow' modes. In the case of D3-branes there is a manifest $SO(2) \times SO(6)$ symmetry that comes from rotations in the (x^1, x^2) and $(x^3, x^6, x^7, x^8, x^9, x^{10})$ planes respectively. In the case of M2-branes I showed that there is an $SO(2) \times SO(2) \times SO(6)$ symmetry corresponding to rotations in the (x^1, x^2) and (x^3, x^4) and $(x^5, x^6, x^7, x^8, x^9, x^{10})$ planes respectively. This enhancement of the R-symmetry from $SO(2) \times SO(6)$ to $SO(2) \times SO(2) \times SO(6)$ presumably comes from taking the strong coupling limit corresponding to the lift to M-theory. Therefore one expects it to be present in the strong coupling DLCQ description of D3-branes but only in the case where $R_3 = R_4$.

Perhaps a more direct relation between the two descriptions can been seen as follows. One is free to compactify $\mathbb{R}^2$ to a torus $\mathbb{T}^2_{12}$. This M2-brane description then becomes motion on $\mathcal{H}_K(su(N), \mathbb{T}^2_{12})$ and the $SO_L(2) \times SO_R(2) \times SO_R(6)$ symmetry is broken to $SO_R(2) \times SO_R(6)$. If one reduces to string theory on x^5 one again obtains the intersecting D2-branes discussed above but one can now T-dualise along x^1, x^2, x^3, x^4 and then lift back to M2-branes. This has the effect of simply swapping the original N M2-branes that were tangent to x^0, x^1, x^2 with the K intersecting M2-branes that were tangent to x^0, x^3, x^4. The result is motion on $\mathcal{H}_N(su(K), \hat{\mathbb{T}}^2_{12})$ where $\hat{\mathbb{T}}^2_{12}$ is the T-dual torus to $\mathbb{T}^2_{12}$. This is almost in agreement with the DLCQ description if we identify $\hat{\mathbb{T}}^2$ with $\hat{\mathbb{T}}^2_{12}$. However there is one caveat: here only the $su(K)$ Lie-algebra and not $u(K)$ is seen. One assumes that this came about because of the gauge group of the three-algebra associated with maximal supersymmetry is $su(2) \oplus su(2)$ rather than $u(N) \oplus u(N)$ that arises in the ABJM model. Thus it would seem that the T-duality and U-duality discussed above manifest themselves as a rank-charge duality in the quantum mechanics on the Hitchin moduli space.

Lastly I examine the formula for $\mathcal{W}$ in the case that was considered in Section 3.3.3 and propose an interpretation for it as the M5-brane momentum $\mathcal{P}_-$. It is known that there are no finite action regular solutions to the Hitchin system on $\mathbb{R}^2$ [155] (more recently see [156]) but here I will make a proposal on how to interpret certain multi-valued solutions. Restoring the factor of l, identifying $\langle A, B\rangle = -\frac{1}{2}\mathrm{tr}(\mathbf{AB})$ (valid in the case considered in Section 3.3.3) and replacing the integral over x^3, x^4, x^- by the volume factor $V_3 = (2\pi)^3 R_3 R_4 R_-$ that would be obtained by taking x^3, x^4, x^- to be periodic one has

$$\mathcal{W} = \frac{\pi}{2l^6} V_3 \frac{i}{2} \int d\mathrm{tr}(\bar{\mathbf{Z}}\mathbf{D}\mathbf{Z}dz) - d\mathrm{tr}(\mathbf{Z}\bar{\mathbf{D}}\bar{\mathbf{Z}}d\bar{z}) \ . \tag{3.4.1}$$

For a smooth solution the integral is only over the sphere at infinity. I assume that for large z one can treat $\mathbf{Z}$ as abelian and ignore $\mathbf{A}$ (which can either be subleading or simply commuting with $\mathbf{Z}$). Then up to a gauge transformation one can expand

$$\mathbf{Z} = -ia\mathbf{J}_3 \ln z + \mathbf{C} + \dots \ , \tag{3.4.2}$$

where $\mathbf{J}_3$ is a real anti-hermitian generator of $so(3)$ normalised to $\mathrm{tr}(\mathbf{J}_3^2) = -2$ and the ellipsis denotes subleading terms. I have assumed this asymptotic form so that $\mathcal{W} \neq 0$. Even so the expression for $\mathcal{W}$ is problematic as there is a divergence:

$$\mathcal{W} = -\frac{\pi i}{4l^6} V_3 \left[\oint 2|a|^2 \frac{\ln \bar{z}}{z} dz + i\mathrm{tr}(a\mathbf{J}_3\bar{\mathbf{C}}) \oint \frac{1}{z} dz \right] + c.c. \ . \tag{3.4.3}$$

However if one cuts-off the divergent terms at some large by finite $r = |z|$ they become

$$
\begin{aligned}
\mathcal{W}_\infty &= -\frac{\pi i}{4l^6} V_3 a|^2 \oint \frac{\ln \bar{z}}{z} dz + c.c. \\
&= -\frac{\pi i}{4l^6} V_3 |a|^2 \oint \ln \bar{z} d \ln z + c.c. \\
&= -\frac{\pi i}{4l^6} V_3 |a|^2 \int_{\ln r - i\pi}^{\ln r + i\pi} \bar{w} dw + c.c. \\
&= \frac{\pi}{4l^6} V_3 |a|^2 \int_{-\pi}^{\pi} (\ln r - i\theta) d\theta + c.c. \\
&= \frac{\pi^2}{l^6} V_3 |a|^2 \ln r \ ,
\end{aligned}
\tag{3.4.4}
$$

where I have introduced a branch cut for $\ln z$ that runs along the negative real axis and written $w = \ln r + i\theta$. Therefore I find

$$
\mathcal{W} = \mathcal{W}_\infty + \frac{\pi^2 i}{2l^6} V_3 \, \mathrm{tr}(\mathbf{J}_3(a\bar{\mathbf{C}} - \bar{a}\mathbf{C})) \ . \tag{3.4.5}
$$

Next observe that $\mathbf{Z}$ is not single valued: under a rotation $z \to e^{2\pi i} z$ one sees that $\mathbf{Z} \cong \mathbf{Z} + 2\pi a \mathbf{J}_3$. Recall that $\mathbf{Z} = \mathbf{Y}^4 + i\mathbf{Y}^3$ where $\mathbf{Y}^4$ and $\mathbf{Y}^3$ are real anti-symmetric matrices. These have imaginary eigenvalues y_4 and y_3 respectively which, after multiplication by i, can be thought of as positions of the two M5-branes along x^4, x^3 directions. The above identification then implies that $y^4 \cong y^4 + 2\pi \mathrm{Re} a$ and $y^3 \cong y^3 + 2\pi \mathrm{Im} a$. One learns from this that Y^3 and Y^4 must be treated as periodic and hence I identify $a = R_4 + iR_3$.

This means that the divergent term only depends on R_3, R_4, R_-. Unfortunately a physical interpretation for this divergence is unclear, it would be interesting to find one. However in this discussion I only want to consider solutions that correspond to fixed radii and so I will simply ignore the divergence and consider instead

$$
\mathcal{W}_{finite} = \frac{\pi^2 i}{l^6} V_3 \, \mathrm{tr}(\mathbf{J}_3(a\bar{\mathbf{C}} - \bar{a}\mathbf{C})) \ . \tag{3.4.6}
$$

I write $\mathbf{C} = c\mathbf{J}_3 + ...$ where the ellipsis denotes terms that are orthogonal to $\mathbf{J}_3$. Thus

$$
\mathcal{W}_{finite} = -\frac{2\pi^2 i}{l^6} V_3 (a\bar{c} - \bar{a}c) \ . \tag{3.4.7}
$$

The multivalued nature of $\mathbf{Z}$ also means that in the space of solutions, those which differ by $c \to c + 2\pi a$ must be identified with each other. Therefore if one writes

$$c = 2\pi R_4 n_4 + 2\pi i R_3 n_3 \ , \tag{3.4.8}$$

then solutions that differ by $(n_3, n_4) \to (n_3+1, n_4+1)$ are identified with each other. As a result I have

$$\begin{aligned} \mathcal{W}_{finite} &= \frac{8\pi^3}{l^6} V_3 R_3 R_4 (n_4 - n_3) \\ &= \left(\frac{V_3}{l^3}\right)^2 \frac{n_4 - n_3}{R_-} \ . \end{aligned} \tag{3.4.9}$$

This suggests that one should identify $l^3 = V_3 = (2\pi)^3 R_3 R_4 R_-$ and so recover the KK spectrum of a null compactification on x^-, provided that $n_4 - n_3$ is an integer. Putting this another way: in order to arrive at the interpretation of this model as describing a null compactification of M5-branes one should assume (Y^3, Y^4) are periodic and impose on the Hitchin system the boundary condition $\mathbf{Z} \sim -i(R_4 + iR_3)\mathbf{J}_3 lnz + 2\pi(R_4 n_4 + iR_3 n_3)\mathbf{J}_3$ where $n_4 - n_3$ is an integer. Lastly I mention that, according to the previous discussion, one is ultimately required to let $R_3, R_4, R_5 \to 0$. However when viewed as the limit of a null boost, the spacelike radius is sent to zero in such that a way that R_- is fixed. In this case $\mathcal{W}_{finite}$ remains finite.

3.5 Summary and Comments

In this chapter I presented a solution to the constraints of the $(2,0)$ system derived in [147]. The result was a system of equations for 3-algebra valued fields $Z, H, X^I, \Psi_\pm$, along with an associated gauge field one-form A, that is defined on a plane $\mathbb{R}^2$ times a null direction $\mathbb{R}_+$ which is used as 'time'. I showed that for choices of the fields X^I, H, Ψ_- that preserve the $\mathcal{Q}_+$ supersymmetries the system reduced to supersymmetric dynamics (with supersymmetry generator $\mathcal{Q}_+$) on the moduli space of an $SO(3)$ Hitchin system. I also gave a physical interpretation of the resulting system as a re-formulation of the M5-brane on $\mathbb{T}^2 \times S^1_-$ to intersecting null M2-branes or alternatively a DLCQ of four-dimensional maximally supersymmetric Yang-Mills.

The original Hitchin system arises in this system for one particular choice of background. In addition these equations admit generalisations such as a non-zero Z_4 and non-constant X^I. It would be interesting to examine these backgrounds and their associated dynamics. It is also possible to include impurities given by sources in the Hitchin equations as done in [152, 153]. One would also expect that these results can be naturally extended to a Lorentzian 3-algebra and hence to an arbitrary gauge group. Finally I also note that Hitchin's system has also appeared before in conjunction with 'so-called' class-S theories derived from the M5-brane [157, 158, 159, 160, 161].

In the next chapter this system will be studied in conjunction with that described in Section 2.3.1 describing M5-branes along a null direction. Actions will be found for these systems.

Chapter 4

Systems of M-branes and Maximally Supersymmetric Non Lorentzian Theories

In this chapter I present gauge theories in $(2+1)$ and $(4+1)$ dimensions with 16 supersymmetries which are invariant under rotations and translations but not boosts. These non-Lorentzian, Galilean invariant theories come from the realisation that the system of null M2-branes from the previous chapter, and the system described in Section 2.3.1 have supersymmetric actions associated to them. This work was conducted with Neil Lambert and was published in JHEP in October 2018 [11].

4.1 Introduction

As discussed earlier, of particular interest of study is the elusive six-dimensional $(2,0)$ theory of N M5-branes [132, 142]. For a variety of reasons it is believed that there is no six-dimensional diffeomorphism invariant lagrangian formulation of the $(2,0)$ theory (see, for example, [134]). However there are a myriad of lagrangians that are associated to lower-dimensional compactifications that can capture some, or even all, of the $(2,0)$ theory dynamics. In particular when reduced on a circle of radius R the $(2,0)$-Theory becomes five-dimensional maximally supersymmetric Yang-Mills theory with gauge group $U(N)$ and coupling $g^2_{YM} = 4\pi^2 R$. Alternatively one can think of the $(2,0)$ theory as providing a strong coupling, UV completion of the perturbatively non-renormalizable five-dimensional Yang-Mills theory [162].

As discussed in Section 2.3.1 and 2.3.2 a non-abelian system of equations was formulated which provide a representation of the six-dimensional $(2,0)$ superalgebra. The system involves a set of dynamical equations as well as some constraint equations. Solving the constraints in different ways leads to maximally supersymmetric Chern-Simons theory in $(2+1)$ dimensions, BLG theory, or maximally supersymmetric Yang-Mills in $(4+1)$ dimensions, corresponding to M2-branes and M5-branes on S^1 respectively.

Owing to the manifest Lorentz symmetry of the system there is also the possibility to construct limits of M2-branes and M5-branes which have been infinitely boosted along some direction (off the brane for M2's but on the brane for M5's). These equations were analysed in [9, 146] for M2-branes and M5-branes respectively and shown to reduce to motion on a moduli space of solitons. In the latter case this reproduces the DLCQ description of the $(2,0)$ theory as motion on the moduli space of self-dual gauge fields [150, 151].

The main goal of this chapter is to show that for these null cases one can construct lagrangians for the dynamics. The results are novel field theories in $(2+1)$ and $(4+1)$ dimensions with 16 supersymmetries, translations and spatial rotations but which are not invariant under boosts. However the field content includes non-dynamical lagrange multiplier fields which restrict the dynamics to motion on a moduli space of solitons. These appear to be a new type of maximally supersymmetric lagrangian.

In Section 4.2 I will study the M2-brane example of Chapter 3. This is a field theory in $(2+1)$ dimensions with maximal supersymmetry, an $SO(2)$ rotational symmetry and an $SO(6)$ R-symmetry. In Section 4.3 I construct the M5-brane example from Section 2.3.1 which is a field theory in $(4+1)$ dimensions with maximal supersymmetry, an $SO(4)$ rotational symmetry and $SO(5)$ R-symmetry. I also briefly explore the dimensional reduction of these theories. In Section 4.4 I comment on how the 8 supersymmetries that arise in the moduli space dynamics are enhanced to 16 supersymmetries in the field theory. In the final section I give a summary and comments.

4.2 An Action for the Null M2

In Chapter 3 a novel system of equations was derived from the system of Section 2.3.2.

Following this analysis one can employ some trial and error to see that the equations of motion, (3.2.11), (3.2.12), (3.2.13), and (3.2.14) arise from the action[1]

$$S_{M2} = S_{scalar} + S_{CS} + S_{fermion} \ , \tag{4.2.1}$$

where

$$\begin{aligned}
S_{scalar} &= \int d^2x dx^+ \Big[\langle D_+ Z, D_+ \bar{Z}\rangle - \langle DX^I, \bar{D}X^I\rangle + \langle D\bar{Z}, \bar{H}\rangle + \langle \bar{D}Z, H\rangle \\
&\qquad -i\langle D_+X^I, [Z,\bar{Z},X^I]\rangle - \frac{1}{2}\langle [X^I,X^J,Z][X^I,X^J,\bar{Z}]\rangle\Big] \\
S_{CS} &= i\int d^2x dx^+ \left[\frac{1}{2}(A_+, F_{z\bar{z}}) + \frac{1}{2}(A_z, F_{\bar{z}+}) + \frac{1}{2}(A_{\bar{z}}, F_{+z}) + \frac{1}{2}(A_+, [A_z, A_{\bar{z}}])\right] \\
S_{fermion} &= \int d^2x dx^+ \Big[\frac{i}{2\sqrt{2}}\langle \Psi_+^T, D_+\Psi_+\rangle + i\langle \Psi_+^T, \hat{\Gamma}_z \bar{D}\Psi_- + \hat{\Gamma}_{\bar{z}} D\Psi_-\rangle \\
&\qquad - \frac{1}{2\sqrt{2}}\langle \Psi_+^T, \hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^{IJ}\left[X^I, X^J, \Psi_+\right]\rangle + \frac{1}{\sqrt{2}}\langle \Psi_-^T, \left[Z, \bar{Z}, \Psi_-\right]\rangle \\
&\qquad + i\langle \Psi_+^T, \hat{\Gamma}^I\hat{\Gamma}_Z\left[Z, X^I, \Psi_-\right]\rangle + i\langle \Psi_+^T, \hat{\Gamma}^I\hat{\Gamma}_{\bar{Z}}\left[\bar{Z}, X^I, \Psi_-\right]\rangle\Big] \ .
\end{aligned} \tag{4.2.2}$$

I explicitly note that the fermions, $\Psi_\pm$, satisfy

$$\hat{\Gamma}_{+12}\Psi_\pm = -\Psi_\pm \qquad \hat{\Gamma}_{+34}\Psi_\pm = \pm\Psi_\pm \ . \tag{4.2.3}$$

where $\hat{\Gamma}_+, \hat{\Gamma}_1, ..., \hat{\Gamma}_{10}$ form a real basis of the $Spin(1,10)$ Clifford algebra.

This action is invariant under the relevant supersymmetries derived in the previ-

[1] Note that compared to Chapter 3 I have rescaled $X^I \to l^{-3/2}X^I, Z \to 2l^{3/2}Z, H \to \frac{1}{2}l^{-3/2}H, \Psi_\pm \to l^{-3/2}\Psi_\pm$ so that the fields have canonical scaling dimensions.

ous chapter, for completeness I include them below in the new normalisation

$$
\begin{aligned}
\delta X^I =& i\epsilon_+^T\hat{\Gamma}^I\Psi_- + i\epsilon_-^T\hat{\Gamma}^I\Psi_+ \\
\delta Z =& \sqrt{2}\epsilon_+^T\hat{\Gamma}_{\bar{Z}}\Psi_+ \\
\delta \bar{Z} =& -\sqrt{2}\epsilon_+^T\hat{\Gamma}_Z\Psi_+ \\
\delta A_z(\,\cdot\,) =& \sqrt{2}\epsilon_+^T\hat{\Gamma}^I\hat{\Gamma}_z[X^I,\Psi_+,\,\cdot\,] + 2i\epsilon_-^T\hat{\Gamma}_z\hat{\Gamma}_{\bar{Z}}\left[\bar{Z},\Psi_+,\,\cdot\,\right] - 2i\epsilon_+^T\hat{\Gamma}_z\hat{\Gamma}_Z\left[Z,\Psi_-,\,\cdot\,\right] \\
\delta A_{\bar{z}} =& -\sqrt{2}\epsilon_+^T\hat{\Gamma}^I\hat{\Gamma}_{\bar{z}}[X^I,\Psi_+,\cdot] + 2i\epsilon_-^T\hat{\Gamma}_{\bar{z}}\hat{\Gamma}_Z\left[Z,\Psi_+,\,\cdot\,\right] - 2i\epsilon_+^T\hat{\Gamma}_{\bar{z}}\hat{\Gamma}_{\bar{Z}}\left[\bar{Z},\Psi_-,\,\cdot\,\right] \\
\delta A_+(\,\cdot\,) =& 2\sqrt{2}i\epsilon_-^T\hat{\Gamma}_Z\left[Z,\Psi_-,\,\cdot\,\right] + 2\sqrt{2}i\epsilon_-^T\hat{\Gamma}_{\bar{Z}}\left[\bar{Z},\Psi_-,\,\cdot\,\right] \\
&+ 2\epsilon_-^T\hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^I\left[X^I,\Psi_+,\,\cdot\,\right] - 2\epsilon_+^T\hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^I\left[X^I,\Psi_-,\,\cdot\,\right] \\
\delta\Psi_+ =& 2i\sqrt{2}\hat{\Gamma}^I\left[Z,\bar{Z},X^I\right]\epsilon_- - 2i\left(\hat{\Gamma}_Z D_+ Z - \hat{\Gamma}_{\bar{Z}} D_+\bar{Z}\right)\epsilon_+ \\
&- \left(\hat{\Gamma}_Z\hat{\Gamma}^{IJ}\left[Z,X^I,X^J\right] + \hat{\Gamma}_{\bar{Z}}\hat{\Gamma}^{IJ}\left[\bar{Z},X^I,X^J\right]\right)\epsilon_+ \\
&+ 2\left(\hat{\Gamma}_{\bar{z}}\hat{\Gamma}^I DX^I + \hat{\Gamma}_z\hat{\Gamma}^I\bar{D}X^I\right)\epsilon_+ \\
&+ 2\sqrt{2}i\left(\hat{\Gamma}_{\bar{z}}\hat{\Gamma}_Z DZ - \hat{\Gamma}_z\hat{\Gamma}_{\bar{Z}}\bar{D}\bar{Z}\right)\epsilon_- \\
\delta\Psi_- =& -\sqrt{2}\hat{\Gamma}^I D_+ X^I\epsilon_+ - \frac{\sqrt{2}i}{3}\hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^{IJK}\left[X^I,X^J,X^K\right]\epsilon_+ \\
&+ \left(\hat{\Gamma}_Z\hat{\Gamma}^{IJ}\left[Z,X^I,X^J\right] + \hat{\Gamma}_{\bar{Z}}\hat{\Gamma}^{IJ}\left[\bar{Z},X^I,X^J\right]\right)\epsilon_- \\
&+ 2\left(\hat{\Gamma}_{\bar{z}}\hat{\Gamma}^I DX^I + \hat{\Gamma}_z\hat{\Gamma}^I\bar{D}X^I\right)\epsilon_- \\
&- 2i\left(\hat{\Gamma}_Z D_+ Z - \hat{\Gamma}_{\bar{Z}} D_+\bar{Z}\right)\epsilon_- + \sqrt{2}i\left(\hat{\Gamma}_{\bar{z}}\hat{\Gamma}_{\bar{Z}} H - \hat{\Gamma}_z\hat{\Gamma}_Z\bar{H}\right)\epsilon_+\ . \\
\delta H =& 2\sqrt{2}\epsilon_-^T\hat{\Gamma}_Z D\Psi_- + 2\epsilon_+^T\hat{\Gamma}_z\hat{\Gamma}_Z D_+\Psi_- \\
&+ i\epsilon_+^T\hat{\Gamma}_z\hat{\Gamma}_Z\hat{\Gamma}^{IJ}\left[X^I,X^J,\Psi_-\right] - 2\sqrt{2}\epsilon_-^T\hat{\Gamma}_z\hat{\Gamma}^I[\bar{Z},X^I,\Psi_-]\ ,
\end{aligned}
\tag{4.2.4}
$$

where

$$
\hat{\Gamma}_{+12}\epsilon_\pm = \epsilon_\pm \qquad \hat{\Gamma}_{+34}\epsilon_\pm = \pm\epsilon_\pm\ . \tag{4.2.5}
$$

While examining the cubic fermion terms that arise in δS it is helpful to observe that they take the same form as the cubic fermion terms that arise in the case of the maximally supersymmetric Lorentzian M2-brane theory (see the appendix).

The action (4.2.1) has some non-standard features. Firstly although the scalars Z have canonical kinetic terms they do not have gradient terms. The scalars X^I have the opposite: no kinetic terms but canonical gradient terms. Furthermore there is a term which is linear in the X^I time-derivative.

One sees that the field H imposes the holomorphic constraint

$$\bar{D}Z = 0 \ . \tag{4.2.6}$$

One also has the Gauss law constraint arising from the A_+ equation of motion:

$$F_{z\bar{z}}(\cdot) = -i\left(\left[Z, D_+\bar{Z},\ \cdot\ \right] + \left[\bar{Z}, D_+Z,\ \cdot\ \right]\right) - \left[X^I, \left[Z, \bar{Z}, X^I\right],\ \cdot\ \right] - \frac{1}{2\sqrt{2}}\left[\Psi_+^T, \Psi_+,\ \cdot\ \right] \ . \tag{4.2.7}$$

For static bosonic configurations these constraints reduce to a 3-algebra form of the Hitchin System:

$$\begin{aligned} \bar{D}Z &= 0 \\ F_{z\bar{z}}(\cdot) &= -\left[X^I, \left[Z, \bar{Z}, X^I\right],\ \cdot\ \right] \ . \end{aligned} \tag{4.2.8}$$

As described before these arise as BPS solutions to the M2-brane [163]. It was shown that allowing for time evolution the dynamical evolution is still restricted to the Hitchin moduli space (at least for a class of configurations). Furthermore this system was identified as describing intersecting M2-branes along the x^1, x^2 and x^3, x^4 directions, in the limit of an infinite boost along x^5.

4.3 Null M5-branes

I now turn attention to a similar construction that represents M5-branes which arises from a null reduction of the model in Section 2.3.1, described in [146]. Although the system is also derived from the 3-algebra construction of [145] it turns out that the resulting dynamical equations can be extended to any gauge group (for example by considering a non-positive definite three-algebra). In particular the field content consists of five scalars X^I (where now $I = 6, 7, 8, 9, 10$), a gauge field one-form (A_+, A_i), $i = 1, 2, 3, 4$ and fermions Ψ all taking values in some Lie-algebra. There is also an anti-self-dual tensor G_{ij}. I consider the action

$$S_{M5} = \frac{1}{g^2}\mathrm{tr}\int d^4x dx^+ \left(\frac{1}{2}F_{+i}F_{+i} - \frac{1}{2}D_iX^ID_iX^I + \frac{1}{2}F_{ij}G_{ij} + \frac{i}{2}\bar{\Psi}\Gamma_-D_+\Psi + \frac{i}{2}\bar{\Psi}\Gamma_iD_i\Psi - \frac{1}{2}\bar{\Psi}[X^I, \Gamma_-\Gamma^I\Psi]\right) , \quad (4.3.1)$$

where $\bar{\Psi} = \Psi^T\Gamma_+$. Here the fermions satisfy $\Gamma_{+12345}\Psi = -\Psi$ and I define

$$\Gamma_\pm = \frac{1}{\sqrt{2}}(\Gamma_5 \pm \Gamma_0) . \quad (4.3.2)$$

Again $\Gamma_+, \Gamma_1, ..., \Gamma_{10}$ are a real representation of the $Spin(1,10)$ Clifford algebra. Note that, unlike the gauge field strength F_{ij}, G_{ij} does not satisfy a Bianchi identity.

The equations of motion arising from this action agree with those found in [146][2]. There it was interpreted as the limit of an infinite boost of M5-branes along x^5. In particular one sees that G_{ij} acts as a Lagrange multiplier imposing self-duality of the spatial components of the gauge field strength; $F_{ij} = \frac{1}{2}\varepsilon_{ijkl}F_{kl}$. Thus the on-shell condition reduces to motion on the moduli space of self-dual gauge fields. In particular the action reduces to a sigma-model on ADHM moduli space which includes a potential and background gauge field that arise from the vacuum expectation values of X^I and A_0 respectively [146].

The on-shell supersymmetries of the system are:

$$\begin{aligned}
\delta X^I &= i\bar{\epsilon}\Gamma^I\Psi \\
\delta A_i &= i\bar{\epsilon}\Gamma_i\Gamma_-\Psi \\
\delta A_+ &= i\bar{\epsilon}\Gamma_{+-}\Psi \\
\delta\Psi &= \Gamma_-\Gamma^ID_+X^I\epsilon + \Gamma_i\Gamma^ID_iX^I\epsilon + \Gamma_i\Gamma_{+-}F_{+i}\epsilon - \frac{1}{4}\Gamma_+\Gamma_{ij}F_{ij}\epsilon \\
&\quad -\frac{1}{4}\Gamma_-\Gamma_{ij}G_{ij}\epsilon - \frac{i}{2}\Gamma_-\Gamma^{IJ}[X^I, X^J]\epsilon \\
\delta G_{ij} &= i\bar{\epsilon}\Gamma_{ij}D_+\Psi + 2i\bar{\epsilon}\Gamma_+\Gamma_{[i}D_{j]}\Psi - \bar{\epsilon}\Gamma_{ij}\Gamma_{+-}\Gamma^I[X^I, \Psi] ,
\end{aligned} \quad (4.3.3)$$

where $\Gamma_{+12345}\epsilon = \epsilon$. These transformations close on-shell and one can check that they leave the action invariant. When checking the vanishing of the cubic fermion terms in δS is it helpful to observe that they have a similar structure to those that

[2]Here we have rescaled the fields from those of reference [145] to their canonical form and also switched the roles of x^+ and x^-.

arise in maximally supersymmetric five-dimensional Yang-Mills (see the appendix).

However one should be careful to ensure that δG_{ij} is anti-self-dual off-shell. To achieve this I note that the transformations (4.3.3) require some modification. First observe that we are free to modify δG_{ij} by

$$\delta G_{ij} \to \delta G_{ij} + i\bar{\epsilon}\Xi\Gamma_{ijk}D_k\Psi \ , \tag{4.3.4}$$

for any choice of Ξ, because the Bianchi identity of F_{ij} ensures that the change in δS is a boundary term. In particular taking $\Xi = \frac{3}{2}\Gamma_+$ I find that

$$\delta G_{ij} + \star\delta G_{ij} = 2\bar{\epsilon}\Gamma_+\Gamma_{ij}E_\Psi \ , \tag{4.3.5}$$

where

$$E_\Psi = i\Gamma_- D_+\Psi + i\Gamma_k D_k\Psi - \Gamma_-\Gamma^I[X^I,\Psi] \ . \tag{4.3.6}$$

is the fermion equation of motion. One can correct this by making the following shift in the supersymmetry transformations:

$$\begin{aligned} \delta G_{ij} &\to \delta G_{ij} - \bar{\epsilon}\Gamma_+\Gamma_{ij}E_\Psi \\ \delta\bar{\Psi} &\to \delta\bar{\Psi} + \frac{1}{2}\bar{\epsilon}\Gamma_+\Gamma_{ij}F_{ij} \ , \end{aligned} \tag{4.3.7}$$

so that the action remains invariant but now δG_{ij} is anti-self dual off-shell.

Thus one concludes that the action (4.3.1) is invariant under the following supersymmetry

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I\Psi \\ \delta A_i &= i\bar{\epsilon}\Gamma_i\Gamma_-\Psi \\ \delta A_+ &= i\bar{\epsilon}\Gamma_{+-}\Psi \\ \delta\Psi &= \Gamma_-\Gamma^I D_+X^I\epsilon + \Gamma_i\Gamma^I D_iX^I\epsilon + \Gamma_i\Gamma_{+-}F_{+i}\epsilon + \frac{1}{4}\Gamma_+\Gamma_{ij}F_{ij}\epsilon \\ &\quad -\frac{1}{4}\Gamma_-\Gamma_{ij}G_{ij}\epsilon - \frac{i}{2}\Gamma_-\Gamma^{IJ}[X^I,X^J]\epsilon \\ \delta G_{ij} &= -\frac{i}{2}\bar{\epsilon}\Gamma_-\Gamma_{ij}\Gamma_+D_+\Psi - \frac{1}{2}\bar{\epsilon}\Gamma_-\Gamma_{ij}\Gamma_+\Gamma^I[X^I,\Psi] \ . \end{aligned} \tag{4.3.8}$$

Lastly I note that one is always free to add an $F_{ij}F_{ij}$ term into the action:

$$S \to S - \frac{\xi}{4g^2}\int d^4x dt\ F_{ij}F_{ij}\ , \tag{4.3.9}$$

for any choice of ξ. This will not change the equations of motion since $D^iF_{ij} = 0$ as a result of the self-dual condition imposed by G_{ij} along with the Bianchi identity of F_{ij}. Furthermore, to preserve supersymmetry, simply shift the variation δG_{ij} to

$$\delta G_{ij} \to \delta G_{ij} + 2i\xi\bar{\epsilon}\Gamma_-\Gamma_{[i}D_{j]}\Psi\ , \tag{4.3.10}$$

so as to ensure $\delta S = 0$. However in the rest of this chapter I will set $\xi = 0$ since on-shell $\xi \neq 0$ leads to infinite contributions to S arising from the integral over time of a constant instanton number.

4.3.1 Dimensional Reduction

The action (4.3.1) provides a non-Lorentz invariant field theory in $(4+1)$ dimensions which is invariant under sixteen supersymmetries, an $ISO(4)$ Euclidean group and an $SO(5)$ R-symmetry. Its on-shell conditions reduce to motion on the moduli space of self-dual gauge fields on $\mathbb{R}^4$ with x^+ playing the role of time.

Clearly one can dimensionally reduce this action to obtain similar ones in $(d+1)$ dimensions with $d < 4$. Following the usual rules of dimensional reduction over $4-d$ dimensions the bosonic field content is now

$$(A_+, A_i) \qquad (X^a = A_{d+1}, .., A_4) \qquad (X^I) \qquad (G_{ij}, G_{ia}, G_{ab})\ , \tag{4.3.11}$$

where now the i index has been reduced to $i = 1, .., d$ with $a = d+1, .., 4$ and as before I have $I = 6, 7, 8, 9, 10$. Note also that anti-self-duality implies that the various components (G_{ij}, G_{ia}, G_{ab}) are not independent. In all these cases the on-shell conditions imply that the dynamics corresponds to motion on the moduli space of self-dual connections reduced to $\mathbb{R}^{4-d}$.

One readily sees from (4.3.1) that scalars X^a will have kinetic terms but X^I will not. Furthermore there will be a potential of the form

$$V \sim -\mathrm{tr}([X^a, X^I][X^a, X^I])\ , \tag{4.3.12}$$

but no potential terms with only X^I or X^a. Thus, unlike the dimensional reduction of Lorentzian maximally supersymmetric Yang-Mills theories, the R-Symmetry is not enhanced to $SO(9-d)$. Rather, upon reduction to $d+1$ dimensions, one finds a maximally super-symmetric field theory with $ISO(d)$ Euclidean symmetry and a $SO(4-d)\times SO(5)$ R-symmetry.

For the sake of completeness I list the dimensional reductions.

Reduction to 3+1 Dimensions

Reduction to 3+1 dimensions one has $(i,j=1,2,3)$

$$(A_+, A_i) \qquad (X^4 \equiv A_4) \qquad (X^I) \qquad (G_{ij}, G_{i4}) \ . \tag{4.3.13}$$

However, since G is anti-self-dual one has the relationship

$$G_{ij} = -\varepsilon_{ijk}G_{k4} \ . \tag{4.3.14}$$

Thus the action becomes

$$\begin{aligned} S_{3+1} = \frac{1}{g^2}\int d^3x dx^+ \Big[& \frac{1}{2}F_{+i}F_{+i} + \frac{1}{2}D_+X^4D_+X^4 + \frac{1}{2}G_{ij}\left(F_{ij} - \varepsilon_{ijk}D_kX^4\right) \\ & - \frac{1}{2}D_iX^ID_iX^I + \frac{1}{2}[X^4, X^I][X^4, X^I] \\ & + \frac{i}{2}\bar{\Psi}\Gamma_-D_+\Psi + \frac{i}{2}\bar{\Psi}\Gamma_iD_i\Psi + \frac{1}{2}\bar{\Psi}\Gamma_4[X^4,\Psi] - \frac{1}{2}\bar{\Psi}[X^I, \Gamma_-\Gamma^I\Psi]\Big] \ . \end{aligned} \tag{4.3.15}$$

Reduction to 2+1

Next I look at the reduction to $(2+1)$ dimensions and compare the result with the M2-action of Section 4.2. The field content is given by ($i=1,2$, $a=3,4$)

$$(A_+, A_i) \qquad (X^a \equiv A_a) \qquad (X^I) \qquad (G_{ij}, G_{ab}, G_{ia}) \ , \tag{4.3.16}$$

but due to anti-self-duality the components G_{ij} and G_{ab} are related as are the various components of G_{ia}. I introduce the complex coordinates

$$z = x^1 + ix^2 \qquad Z = X^4 + iX^3 \ , \tag{4.3.17}$$

and

$$D = \frac{1}{2}(D_1 - iD_2) \qquad \Gamma_Z = \frac{1}{2}(\Gamma_4 - i\Gamma_3)\ . \tag{4.3.18}$$

I also re-express the independent components of the Lagrange multiplier field as

$$G = G_{12} = -G_{34} \qquad H = G_{14} - iG_{13}\ . \tag{4.3.19}$$

With these definitions one can write the reduced action as

$$\begin{aligned} S_{2+1} \;=\; \frac{1}{g^2}\mathrm{tr}\int d^2x dx^+ \Big(& \frac{1}{2}F_{+z}F_{+\bar{z}} + \frac{1}{2}D_+ZD_+\bar{Z} + \bar{H}D\bar{Z} + H\bar{D}Z \\ & - DX^I\bar{D}X^I + \frac{1}{2}[Z,X^I][\bar{Z},X^I] - 2iG\left(F_{z\bar{z}} - \frac{1}{4}[Z,\bar{Z}]\right) \\ & + \frac{i}{2}\bar{\Psi}\Gamma_-D_+\Psi + i\bar{\Psi}(\Gamma_{\bar{z}}D\Psi + \Gamma_z\bar{D}\Psi) + \frac{1}{2}\bar{\Psi}\Gamma_Z[Z,\Psi] + \frac{1}{2}\bar{\Psi}\Gamma_{\bar{Z}}[\bar{Z},X^I] \\ & - \frac{1}{2}\bar{\Psi}[X^I,\Gamma_-\Gamma^I\Psi]\Big)\ . \end{aligned} \tag{4.3.20}$$

The on-shell conditions now reduce to motion on the moduli space of solutions to the Hitchin System, this time for any gauge group. However, although it has the same number of supersymmetries as the M2-brane case discussed above it only has $SO(2)\times SO(5)$ R-symmetry, not $SO(2)\times SO(6)$. It is natural to postulate that, just as the Lorentzian M2-brane theory is the strong coupling limit of $(2+1)$-dimensional maximally supersymmetric Yang-Mills (which can be viewed as the dimensional reduction of the M5-brane), the null M2-brane theory (4.2.1) is the strong coupling fixed point of the null M5-brane action (4.3.20) in the case of an $SU(2)$ gauge group.

Reduction to 1+1 Dimensions

Next I consider the reduction to 1+1 Dimensions. Here the bosonic fields are $(a = 2,3,4)$

$$(A_+, A_1) \qquad (X^a \equiv A_a) \qquad (X^I) \qquad (G_{ab}, B_a = G_{1a})\ . \tag{4.3.21}$$

However, I am taking that G is anti self-dual so one has the relationship

$$G_{ab} = -\varepsilon_{abc}B_c\ . \tag{4.3.22}$$

The action can be written now as

$$S_{1+1} = \frac{1}{g^2}\int dxdx^+ \left[\frac{1}{2}F_{+1}F_{+1} + \frac{1}{2}D_+X^aD_+X^a - \frac{1}{2}D_1X^ID_1X^I + \frac{1}{2}[X^a,X^I][X^a,X^I] \right.$$
$$-\frac{1}{2}G_{ab}(\varepsilon_{abc}D_1X^c + i[X^a,X^b])$$
$$\left. + \frac{i}{2}\bar{\Psi}\Gamma_-D_+\Psi + \frac{i}{2}\bar{\Psi}\Gamma_1D_1\Psi + \frac{1}{2}\bar{\Psi}\Gamma_a[X^a,\Psi] - \frac{1}{2}\bar{\Psi}[X^I,\Gamma_-\Gamma^I\Psi]\right] . \tag{4.3.23}$$

Here one sees that the Lagrange multiplier reduces the theory to motion on the moduli space of Nahm's equations.

Reduction to 0+1 Dimensions

Lastly one can consider the case of a reduction to 0+1 dimensions. The bosonic fields are ($a = 1,2,3,4$)

$$(A_+) \qquad (X^a \equiv A_a) \qquad (X^I) \qquad (G_{ab}) , \tag{4.3.24}$$

and now G_{ab} is anti-self-dual. The action becomes

$$S_{0+1} = \frac{1}{g^2}\int dx^+ \left[\frac{1}{2}D_+X^aD_+X^a + \frac{1}{2}[X^a,X^I][X^a,X^I] - \frac{i}{2}G_{ab}[X^a,X^b] \right.$$
$$\left. + \frac{i}{2}\bar{\Psi}\Gamma_-D_+\Psi + \frac{1}{2}\bar{\Psi}\Gamma_a[X^a,\Psi] - \frac{1}{2}\bar{\Psi}[X^I,\Gamma_-\Gamma^I\Psi]\right] . \tag{4.3.25}$$

This is itself a quantum mechanical model whose on-shell equations of motion reduce it to a sigma model on the moduli space of matrices that satisfy

$$[X^a,X^b] = \frac{1}{2}\varepsilon^{abcd}[X^c,X^d] . \tag{4.3.26}$$

However there are no finite dimensional non-trivial solutions to this system. To see this one observes that the expression

$$V = -\mathrm{tr}([X^a,X^b][X^a,X^b]) \tag{4.3.27}$$

is positive definite but when evaluated on (4.3.26) we find

$$V = \frac{1}{2}\varepsilon^{abcd}\mathrm{tr}(X^a[X^b,[X^c,X^d]]) , \tag{4.3.28}$$

which vanishes by the Jacobi identity and hence $[X^a, X^b] = 0$. Nevertheless it might be interesting to explore any applications for this model in terms of the Matrix theory approach to M-theory.

4.4 Eight vs Sixteen Supersymmetries

In the examples above I have constructed field theories in a variety of dimensions which are invariant under sixteen supersymmetries. However the on-shell conditions reduce the dynamics to one-dimensional motion on a finite-dimensional moduli space of BPS configurations (self-dual gauge fields and their various dimensional reductions). However these moduli spaces are hyper-Kähler and as such the one-dimensional sigma-models describing their dynamics possess only 8 supersymmetries. What has happened?

To resolve this paradox observe that the the sixteen supersymmetries split into $(\mathcal{Q}_+, \mathcal{Q}_-)$ and their algebra takes the form [9, 146], described above

$$\begin{aligned}
&\{\mathcal{Q}_+, \mathcal{Q}_+\} \sim \mathcal{P}_+ \\
&\{\mathcal{Q}_+, \mathcal{Q}_-\} \sim \mathcal{P} \\
&\{\mathcal{Q}_-, \mathcal{Q}_-\} \sim \mathcal{P}_- \ .
\end{aligned} \tag{4.4.1}$$

Here $\mathcal{P}_+$ is the energy arising from the lagrangians above, $\mathcal{P}$ denote the spatial momenta and $\mathcal{P}_-$ is a topological index, such as the instanton number. In particular this index is, up to an overall scale, integer $\mathcal{P}_- \sim n \in \mathbb{Z}$ and the moduli space of BPS solutions $\mathcal{M}$ is graded by n:

$$\mathcal{M} = \oplus_{n\in\mathbb{Z}}\mathcal{M}_n \ . \tag{4.4.2}$$

Within each component $\mathcal{M}_n$ (apart from $n = 0$) one sees that $\{\mathcal{Q}_-, \mathcal{Q}_-\} \neq 0$ and hence the $\mathcal{Q}_-$ supersymmetries are broken. Thus the resulting moduli space dynamics are only invariant under the eight $\mathcal{Q}_+$ supersymmetries. For $n = 0$ the moduli space is flat and all sixteen supersymmetries are again realised. Thus by embedding these one-dimensional sigma model dynamics in to a field theory it is possible to realise the full 16 supersymmetries and also make their higher-dimensional interpretation more transparent.

4.5 Summary and Comments

In this chapter I have presented non-abelian actions in $(2+1)$ and $(4+1)$ dimensions (along with the dimensional reduction of the latter) without boost invariance. In particular some fields lack kinetic terms. As such one might be concerned that there is nothing to suppress them and the resulting theory will be pathological. However there are also Lagrange multiplier fields that restrict the dynamics to a moduli space of BPS configurations. As a result the kinetic energy of all the fields is controlled and the actions can be reduced to one-dimensional motion on the moduli space. This last step then breaks half of the supersymmetry. One could state this result the other way around: I have embedded one-dimensional moduli space dynamics into a field theory and thereby doubled the supersymmetry and clarified the spacetime interpretation.

The sigma-models that result from these actions are certainly not new. For example for the case of the M5-brane they have appeared as a DLCQ prescription for the M5-brane $(2,0)$ theory [150, 151]. Indeed this result provides another perspective on how this model relates to the $(2,0)$ theory. The AdS dual to these and similar DLCQ models was studied in [164, 165, 166] and it would be interesting to make contact with this analysis.

These actions have been derived by solving the constraints of the $(2,0)$ system of [145, 147] in the special null cases that were studied in [146, 9]. As such they are expected to describe limits of M2-branes and M5-branes where the branes have been infinitely boosted so that their worldvolume time coordinate becomes light-like. In other words in this construction these actions arise as a limit of an infinite boost of static M2-branes and M5-branes, *aka* null M2-branes and M5-branes. Such embeddings were discussed in [167] for the case of single branes. It is amusing to observe that the Lagrange multiplier fields H and G_{ij} which appear in our non-Lorentian actions both arise as components of the self-dual three-form of the six-dimensional $(2,0)$ supermultiplet.

Chapter 5

Chiral Modes From M5-branes

In this chapter I shift focus from the previous two chapters. The D4-D6 system is a well studied and understood system in string theory. A key feature of this configuration is the existence of chiral modes arising along the one dimensional intersection; equivalently this is interpreted as arising from the open strings stretched between the D4 and D6-branes. How these states arise from the M-theory perspective is mysterious as the M-theory equivalent configuration is an M5-brane wrapped on the everywhere smooth multi Taub-NUT space and is thus described by the little understood $(2,0)$ theory. The next chapter will demonstrate a way to understand these chiral modes from M-theory by circumnavigating our difficulties with the M5-branes worldvolume theory. This work was published in JHEP [10] with Neil Lambert in April 2018.

5.1 Introduction

One case where the degrees of freedom of M5-branes seem particularly mysterious is when the $(2,0)$ theory is considered as wrapped on a multi-centred Taub-NUT space $\mathcal{M}_{mTN}$. This is a completely smooth four-dimensional manifold which is the generalisation to multiple cores of the geometry of equation (1.2.7). One expects that the $(2,0)$ theory on $\mathbb{R}^{1,1} \times \mathcal{M}_{mTN}$ is locally the same as on $\mathbb{R}^{1,5}$. On the other hand reducing on the S^1 fibration leads to a string theory picture of N D4-branes intersecting with D6-branes which are localised at the zeros of the $U(1)$ Killing vector of multi-centred Taub-NUT space. From standard D-brane dynamics one finds that there are stretched D4-D6 strings which are localised at these zeros. In particular these are so-called '$DN = 8$ strings' whose ground state consists of chiral fermions which propagate along $\mathbb{R}^{1,1}$ and lie in the bi-fundamental of $U(N) \times U(N_I)$ where N_I is the number of coincident D6-branes located at the I^{th} zero. These fermions have been studied in [86] and [168]. Similar states have also appeared in [169] in the case of M5-branes wrapped on cycles in elliptic Calabi-Yau compactifications. The main question addressed in this chapter is how do such charged states arise from the $(2,0)$ theory?

This question arises even in the case of a single M5-brane, corresponding to $N = 1$, where the M5-brane equations are known. However there is still a puzzle: The chiral fermions are charged under the worldvolume gauge field but none of the fields in the M5-brane theory have a minimal coupling so that their quanta can support a charge. This follows from the fact that for a single M5-brane all the fields have an interpretation as Goldstone modes [170] and hence, by Goldstones theorem, they only have derivative interactions. It will be shown that the resolution of this puzzle is that the chiral fermions arise as soliton states on the M5-brane and Goldstone's theorem does not apply to solitons, *i.e.* Goldstone modes can have non-derivative couplings with solitons [171]. Aspects of this case have appeared in [172] and in Section 5.2 I review this along with some unpublished notes [173].

Thus the chiral modes arise from the same sort of mechanism that appeared in [174]. There the chiral modes of the Heterotic string worldsheet in a $\mathbb{T}^3$ compactification were obtained from zero-modes of the two-form gauge potential obtained in Kaluza-Klein reduction of an M5-brane on $K3$. However there is a key difference here in that there is a gauge field under which the chiral modes are charged.

In the non-abelian case of N M5-branes it was argued in [168] that the D4-D6 strings give rise to an $U(N)$ WZWN model. The main result in this chapter is to derive these states and the associated WZWN model from the $(2,0)$-theory alone, without appealing to a D-brane construction using open strings. In particular I will use a variation of five-dimensional maximally supersymmetric Yang-Mills that was constructed in [175, 176] as the natural non-abelian extension of the abelian $(2,0)$ theory reduced on the circle fibration of $\mathcal{M}_{mTN}$. I present these solitons in Section 5.3 and obtain the WZWN model in Section 5.4. Finally I provide a summary in 5.5.

5.2 The Abelian Case

I start by recalling the linearized equations of motion of a single M5-brane from Section 2.2 which is just that of a six-dimensional abelian tensor multiplet [177]. Since this chapter uses the work of [175] closely, I will work in their notation:

$$\begin{aligned}
\nabla^2\phi^{\alpha}{}_{\beta} &= 0 \\
i\Gamma^m\nabla_m\psi^{\alpha} &= 0 \\
H_{mnp} &= \frac{1}{3!}\epsilon_{mnpqrs}H^{qrs} \ .
\end{aligned} \tag{5.2.1}$$

Here $m, n, p = 0, 1, 2, 3, 4, 5$, $H_{mnp} = 3\partial_{[m}B_{np]}$ and $\epsilon^{012345} = 1$. In addition $\alpha, \beta = 1, 2, 3, 4$ denote indices of the fundamental **4** representation of the R-symmetry group $USp(4)$ which are raised (lowered) with the invariant tensor $M^{\alpha\beta}$ ($M_{\alpha\beta}$) and $\phi^{(\alpha\beta)} = M_{\alpha\beta}\phi^{\alpha\beta} = 0$. These equations are invariant under the supersymmetry transformations

$$\begin{aligned}
\delta\phi^{\alpha\beta} &= -i\bar{\epsilon}^{[\alpha}\psi^{\beta]} \\
\delta B_{mn} &= -i\bar{\epsilon}^{\alpha}\Gamma_{mn}\psi_{\alpha} \\
\delta\psi^{\alpha} &= \nabla_m\phi^{\alpha}{}_{\beta}\Gamma^m\epsilon^{\beta} + \frac{1}{2\cdot 3!}\Gamma^{mnp}H_{mnp}\epsilon^{\alpha} \ ,
\end{aligned} \tag{5.2.2}$$

provided that ϵ^{α} is a chiral Killing spinor on the M5-brane worldvolume: $\nabla_{\mu}\epsilon^{\alpha} = 0$, $\Gamma_{012345}\epsilon^{\alpha} = \epsilon^{\alpha}$ and subject to a reality condition.

In the configuration considered here the M5-brane worldvolume is $\mathbf{R}^{1,1} \times \mathcal{M}_{mTN}$ with metric

$$ds_6^2 = -(dx^0)^2 + (dx^1)^2 + ds^2_{mTN} \ . \tag{5.2.3}$$

Here $\mathcal{M}_{mTN}$ is the n-centred multi-centred Taub-NUT space [178]:

$$ds^2_{mTN} = H^{-1}(dx^5 + \theta)^2 + H d\vec{x}\cdot d\vec{x} \ , \tag{5.2.4}$$

where

$$H = 1 + \sum_{I=1}^{n} h_I \ , \qquad \theta = \sum_{I=1}^{n}\theta_I \ , \tag{5.2.5}$$

and

$$h_I = \frac{R}{2}\frac{N_I}{|\vec{x} - \vec{x}_I|} \ , \qquad d\theta_I = \star_3 dh_I \ . \tag{5.2.6}$$

For $N_I = 1$ the metric is smooth everywhere provided that one makes the identification $x^5 \sim x^5 + 2\pi R$. I have introduced the integer N_I to allow for N_I coincident D6-branes at a given pole $\vec{x}^I$ in the x^7, x^8, x^9 plane. For $N_I > 1$ this induces a conical singularity at the poles. Asymptotically this metric takes the form

$$\begin{aligned} ds^2_{mTN} &= \left(1+\frac{N_{D6}R}{2r}\right)^{-1}\left(dx^5+\frac{1}{2}N_{D6}R\cos\theta d\phi\right)^2 \\ &\quad +\left(1+\frac{N_{D6}R}{2r}\right)\left(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2\right) \ , \end{aligned} \tag{5.2.7}$$

where $N_{D6} = N_1 + ... + N_n$ is the total number of D6-branes. In this case, or for any other manifold $\mathcal{M}$ with self-dual curvature there exists a Killing spinor ϵ^α that satisfies

$$\Gamma_{2345}\epsilon^\alpha = -\epsilon^\alpha \tag{5.2.8}$$

Which is equivalent to the condition $\Gamma_{01}\epsilon^\alpha = -\epsilon^\alpha$.

Next I look for bosonic solutions to the equations of motion which preserve all of these remaining 8 supersymmetries. One cannot impose any more conditions on the Killing spinor and so it must be that $\partial_m \phi^\alpha{}_\beta = 0$. Hence without loss of generality take $\phi^\alpha{}_\beta = 0$. Introducing light cone coordinates

$$x^- = \frac{x^1 - x^0}{\sqrt{2}} \qquad x^+ = \frac{x^1 + x^0}{\sqrt{2}} \ , \tag{5.2.9}$$

one sees that

$$\delta\psi^\alpha = \frac{1}{4}\Gamma^{-ij}H_{-ij}\epsilon^\alpha + \frac{1}{4}\Gamma^{+ij}H_{+ij}\epsilon^\alpha + \frac{1}{2}\Gamma^{+-i}H_{+-i}\epsilon^\alpha + \frac{1}{3!}\Gamma^{ijk}H_{ijk}\epsilon^\alpha = 0 \ , \tag{5.2.10}$$

where $i, j = 2, 3, 4, 5$. Since $\Gamma_-\epsilon^\alpha = \Gamma^+\epsilon^\alpha = 0$, and demanding the remaining 8 supersymmetries be preserved, I find that $H_{-ij} = H_{ijk} = H_{+-i} = 0$ so the solutions to the linearized equation of motion which preserve the $(0, 8)$ supersymmetries are simply

$$H = \sum_{I=1}^{n} \nu_+^I dx^+ \wedge \omega_I \ . \tag{5.2.11}$$

Furthermore self-duality and closure of H implies that the ω_I are self-dual harmonic two-forms on $\mathcal{M}_{mTN}$ whereas the ν_+^I are arbitrary functions of x^+.

Indeed one can explicitly construct n self-dual two-forms on multi-centred Taub-

NUT as in [179]

$$\omega_I = \frac{1}{4\pi^2 R} d\xi_I \ , \qquad \xi_I = H^{-1} h_I (dx^5 + \theta) - \theta_I \ , \tag{5.2.12}$$

where I introduce a useful normalisation to ensure that the ω_I are dimensionless and which will be justified later. These forms are smooth everywhere (at least in the case $N_I = 1$) and satisfy

$$\int \omega_I \wedge \omega_J = \int \omega_I \wedge \star\omega_J = \frac{N_I}{4\pi^2}\delta_{IJ} \ . \tag{5.2.13}$$

One can also see that there are no fermion zero-modes. In particular imposing $\partial_- \psi^\alpha = 0$ one sees that the fermion equation is simply $\Gamma^i \nabla_i \psi^\alpha = 0$ and it is a well-known result that there are no solutions to the Dirac equation which vanish at infinity. Thus the solitons are non-degenerate and do not form an enhanced multiplet of the Lorentz group.

For vanishing scalars and fermions the energy-momentum tensor is simply [180]

$$T_{mn} = \frac{\pi}{2}\sqrt{-g} H_{mpq} H_n{}^{pq} \ . \tag{5.2.14}$$

In which case only T_{++} is non-vanishing and I define

$$\begin{aligned} \mathcal{P}_+ &= \int d^5x \, T_{++} \\ &= \frac{1}{4\pi} \sum N_I \int dx^+ \nu_+^I(x^+) \nu_+^I(x^+) \ . \end{aligned} \tag{5.2.15}$$

In particular the abelian $(2,0)$-theory contains the conserved current (the coefficient is chosen for future convenience)

$$J_m(\Lambda) = 2\pi\sqrt{-g} H_{mnp} \partial^n \Lambda^p \ , \tag{5.2.16}$$

for any choice of one-form Λ inherited from the gauge symmetry $B \to B + d\Lambda$. On-shell the associated charge is a total derivative:

$$\begin{aligned} \mathcal{Q}(\Lambda) &= \int_{\mathbb{R}\times\mathcal{M}_{mTN}} J_+(\Lambda) d^4x dx^+ \\ &= 2\pi \oint_{\mathbb{R}\times S^1\times S^2_\infty} H_{+r\mu}\Lambda^\mu \, r^2 d\Omega_2 dx^+ dx^5 \ , \end{aligned} \tag{5.2.17}$$

where $S^1 \times S^2_\infty$ is the asymptotic form of $\mathcal{M}_{mTN}$, r the radial direction, and Ω_2 the metric on a 2-sphere. Taking only Λ_5 non-vanishing I find

$$\begin{aligned}\mathcal{Q}(\Lambda_5(\infty)) &= \frac{1}{2\pi R}\mathrm{tr}\sum_I \oint_{\mathbb{R}\times S^2_\infty} d\Omega_2 dx^+ \left[H\partial_r\left(\frac{h_I}{H}\right) + \varepsilon^{rjk}\theta_j\partial_k\left(\frac{h_I}{H}\right)\right]\nu^I_+\Lambda_5(\infty) \\ &= -2\pi R\sum_I N_I\int dx^+\nu^I_+(x^+)\Lambda_5(\infty)\ ,\end{aligned} \tag{5.2.18}$$

where the second term in the first line arises as $\Lambda^i = g^{i5}\Lambda_5 \neq 0$. Upon reduction on the S^1 parameterized by x^5 the D4-brane $U(1)$ gauge field is $A_\mu = 4\pi^2 R B_{\mu 5}$ [149] and the $U(1)$ gauge symmetry is $A_\mu \to A_\mu + 4\pi^2 R\partial_\mu\Lambda_5$. Thus $\mathcal{Q}(\Lambda_5(\infty))$ is the corresponding electric charge that one expects and each ν^I_+ carries N_I units of its charge.

5.3 The Non-Abelian Case

In general there is no satisfactory formulation of the M5-brane in the non-abelian case. Nevertheless the M5-brane on a circle of radius R gives, at least at low energy, five-dimensional maximally supersymmetric Yang-Mills. Therefore one can reduce the abelian theory on the S^1 fibration in $\mathcal{M}_{mTN}$ and then find the appropriate non-abelian generalisation. This was done in [175, 176]. I first give their result. Reducing on x^5 leads to the five-dimensional metric

$$ds_5^2 = -(dx^0)^2 + (dx^1)^2 + H d\vec{x}\cdot d\vec{x}\ . \tag{5.3.1}$$

For the discussion here I need the gauge field action which is[1]

$$S_F = \frac{1}{8\pi^2 R}\int d^5x\sqrt{H}\mathrm{tr}(F\wedge\star F) + \theta\wedge\mathrm{tr}(F\wedge F)\ , \tag{5.3.2}$$

where $\mu,\nu = 0,1,2,3,4$. For computing the energy-momentum tensor I will also need the scalar action which is

$$\begin{aligned}S_\phi &= -\frac{1}{8\pi^2 R}\mathrm{tr}\int d^5x\sqrt{-g}\Big(\sqrt{H}D_\mu\phi_{\alpha\beta}D^\mu\phi^{\alpha\beta} + \frac{1}{4}\frac{1}{H^{5/2}}\partial_i H\partial_i H\phi_{\alpha\beta}\phi^{\alpha\beta} \\ &\qquad -\sqrt{H}[\phi^{\alpha\beta},\phi_\beta{}^\delta][\phi_{\delta\gamma},\phi^\gamma{}_\alpha]\Big)\ .\end{aligned} \tag{5.3.3}$$

[1] I use a convention where $\frac{1}{8\pi^2}\mathrm{tr}\int F\wedge F\in\mathbb{Z}$.

Note that I could introduce an alternative form for the gauge part of the action:

$$S'_F = \frac{1}{8\pi^2 R}\int d^5x\sqrt{H}\mathrm{tr}(F\wedge\star F)+\mathcal{F}\wedge CS \ , \tag{5.3.4}$$

where

$$CS = \mathrm{tr}\left(A_\mu\partial_\nu A_\lambda+\frac{2}{3}A_\mu A_\nu A_\lambda\right)dx^\mu\wedge dx^\nu\wedge dx^\lambda \ . \tag{5.3.5}$$

These two actions differ by whether the topological term is taken to be $\theta\wedge\mathrm{tr}(F\wedge F)$ or $\mathcal{F}\wedge CS$. In turn these choices differ by boundary terms arising from the poles of H and infinity and hence have the same equations of motion. The first choice preserves all gauge symmetries of the action but depends upon the choice of θ and hence is not diffeomorphism invariant. Whereas the second form is diffeomorphism invariant but at the expense of introducing potential violations of worldvolume gauge symmetries. I will mainly be interested in the first case, however in Section 5.4 some of the physical differences that arise from the second will be explored, and which rule it out as the correct one. Indeed part of the motivation of this work is to explore such subtleties.

5.3.1 D4-D6 Strings as Solitons

I work from results in [175] which give the five-dimensional theory resulting after reduction over x^5. The prescription for the decomposition from six dimensions to five dimensions is given in the paper and I thus denote the decomposed five-dimensional gamma matrices by γ, and the five-dimensional Killing spinor by ε. One then finds that equation (5.2.8) reduces, after the decomposition, to the condition

$$i\gamma_{234}\varepsilon^\alpha=\varepsilon^\alpha \ , \tag{5.3.6}$$

equivalently

$$\gamma^{01}\varepsilon^\alpha=\varepsilon^\alpha \ . \tag{5.3.7}$$

The fermionic supersymmetry variation from [175] is given by

$$\begin{aligned}\delta\psi^\alpha &= \frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\varepsilon^\alpha+2i\sqrt{H}M_{\beta\gamma}D_\mu\left(\frac{1}{\sqrt{H}}\phi^{\alpha\beta}\right)\gamma^\mu\varepsilon^\gamma \\ &- \frac{1}{\sqrt{H}}M_{\beta\gamma}\phi^{\alpha\beta}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^\gamma+2M_{\beta\gamma}M_{\delta\lambda}[\phi^{\alpha\beta},\phi^{\gamma\delta}]\varepsilon^\lambda \ ,\end{aligned} \tag{5.3.8}$$

with $\mathcal{F} = d\theta$ and I recall that the six-dimensional two-form, $B_{\mu\nu}$, is reduced to a $U(1)$ gauge field as $A_\mu = B_{\mu 5}$ with corresponding field strength

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu] \ , \tag{5.3.9}$$

and thus a gauge covariant derivative defined by

$$D_\mu \chi = \nabla_\mu \chi + [A_\mu, \chi] \ , \tag{5.3.10}$$

where χ is some field transforming in the adjoint of the gauge group.

I seek bosonic, BPS states of the configuration to find those maximally supersymmetric states. This is equivalent to setting equation (5.3.8) to zero. Using the Killing spinor conditions above and after changing to the light cone coordinates introduced in the abelian case, I find that the BPS conditions for this system are

$$F_{ij} = F_{+-} = F_{i-} = 0 \ , \tag{5.3.11}$$

where from now on $i, j = 2, 3, 4$ and also

$$D_i\left(\sqrt{H}\phi^\alpha{}_\beta\right) = D_-\phi^\alpha{}_\beta = 0, \qquad [\phi^\alpha{}_\beta, \phi^\beta{}_\gamma] = 0 \ . \tag{5.3.12}$$

In addition one can compute the equation of motion from the action (5.3.4) and obtain

$$\sqrt{-g}D_\sigma\left(\sqrt{H}F^{\sigma\lambda}\right) + \frac{1}{4}\mathcal{F}_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma\lambda} = 0 \ . \tag{5.3.13}$$

Upon enforcing the BPS conditions above this equation of motion reduces to

$$\partial_i F_{+i} + [A_i, F_{+i}] + 2\partial_i H F_{+i} = 0 \ . \tag{5.3.14}$$

First, looking at (5.3.11), choose to set $A_i = A_- = 0$, then I have that $A_+ = A_+(x^+, x^i)$ solves these conditions.

Now turning to (5.3.12), notice that a solution is given by the ansatz $\phi^\alpha{}_\beta = \frac{1}{\sqrt{H}}\phi^\alpha_0{}_\beta(x^+)$ with the understanding that $[\phi^\alpha_0{}_\beta, \phi^\beta_0{}_\gamma] = 0$.

To solve the equation of motion (5.3.14) I start by noting that the general solution to the BPS conditions $F_{ij} = F_{i-} = 0$ is given by

$$A_i = g\partial_i g^{-1} \qquad A_- = g\partial_- g^{-1} \ , \tag{5.3.15}$$

for a arbitrary element g of the unbroken gauge group. Similarly the solution to BPS condition $F_{+-} = 0$ implies that

$$A_+ = g'\partial_+ g'^{-1} \qquad A_- = g'\partial_- g'^{-1} \ , \tag{5.3.16}$$

for some other element g' of the unbroken gauge group. Consistency of these two expressions for A_- implies that $g'^{-1}g\partial_-(g^{-1}g') = 0$ and hence

$$g' = gk \qquad \text{with} \qquad \partial_- k = 0 \ . \tag{5.3.17}$$

Thus one sees that the generic solution to the BPS equation is simply a gauge transformation by g of the configuration $A_+ = k\partial_+ k^{-1}$, $A_- = A_i = 0$, corresponding to $F_{i+} = \partial_i A_+$.

To continue, fix the gauge $A_- = A_i = 0$ and pick an ansatz for A_+ of the form $A_+ = K(\vec{x})\nu_+(x^+)$ for some $K(\vec{x})$; this means that the equation of motion becomes

$$\partial_i\partial_i K + \frac{2}{H}\partial_i K \partial_i H = 0 \ . \tag{5.3.18}$$

Solutions to this equation are of the form

$$K = \frac{h}{H} \ , \tag{5.3.19}$$

where h is any harmonic function: $\partial_i\partial_i h = 0$. However, one wishes to look for solutions with finite energy. To achieve this, any pole in h must be cancelled by a pole in H (see the expressions below for the energy-momentum tensor) and therefore the solutions are

$$K_I = \frac{h_I}{H} = \frac{h_I}{1 + \sum_J h_J} \ . \tag{5.3.20}$$

One might worry that there is another finite energy solution K_0 corresponding to $h = 1$. However one sees that

$$\sum_I K_I = \frac{H-1}{H} = 1 - K_0 \ . \tag{5.3.21}$$

Rearranging this one see that the solution

$$A_+ = K_0\nu_+^0 + \sum_I K_I\nu_+^I = \nu_+^0 + \nu_+^1 + ... + \nu_+^n \ , \tag{5.3.22}$$

is pure gauge. Therefore I conclude that the most general finite-energy soliton solution is

$$A_+ = \sum_{I=1}^{n} K_I(\vec{x})\nu_+^I(x^+) \ , \tag{5.3.23}$$

where ν_+^I is an arbitrary x^+-dependent element of the unbroken gauge algebra. Of course one can indeed check that these functions K_I also appear in the self-dual two-forms constructed above as $K_I = \xi_{I5}$. In particular these solutions are

$$\begin{aligned} F &= \sum_I \nu_+^I(x^+)\partial_i K_I dx^+ \wedge dx^i \\ &= 4\pi^2 R\sum_I \nu_+^I(x^+)\omega_{i5}^I dx^+ \wedge dx^i \ , \end{aligned} \tag{5.3.24}$$

which corresponds to a simple embedding of the abelian solution into the non-abelian theory by promoting ν_+^I to a element of the unbroken M5-brane gauge algebra and identifying

$$F_{\mu\nu} = 4\pi^2 R H_{\mu\nu 5} \ , \tag{5.3.25}$$

in agreement with [149], explaining the normalization in (5.2.14).

One can also see that there are no fermionic zero-modes. The fermionic equation is [175]

$$i\sqrt{H}\gamma^\mu D_\mu\psi^\alpha - \frac{1}{8}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\psi^\alpha = 0 \ . \tag{5.3.26}$$

Imposing $\partial_-\psi^\alpha = 0$ and expanding around the solitons one finds that this splits into two chiral equations

$$\begin{aligned} -\sqrt{2}\gamma_0 H^{\frac{1}{2}} D_+\psi_+^\alpha + \vec{\gamma}\cdot\vec{\nabla}\psi_-^\alpha + \frac{1}{4}H^{-\frac{1}{2}}\vec{\gamma}\cdot\vec{\nabla}H\psi_-^\alpha &= 0 \\ \vec{\gamma}\cdot\vec{\nabla}\psi_+^\alpha - \frac{1}{4}H^{-\frac{1}{2}}\vec{\gamma}\cdot\vec{\nabla}H\psi_+^\alpha &= 0 \ . \end{aligned} \tag{5.3.27}$$

Note that the only appearance of the non-abelian gauge field is through the D_+ term in the first equation. The second equation is simply the Dirac equation for $\hat{\psi}_+^\alpha = e^{-\frac{1}{2}H^{1/2}}\psi_+^\alpha$, *i.e.* $\vec{\gamma}\cdot\vec{\nabla}\hat{\psi}_+^\alpha = 0$. As with the abelian case there are no solutions which vanish at infinity and hence $\psi_+^\alpha = 0$. In this case the first equation becomes the Dirac equation $\vec{\gamma}\cdot\vec{\nabla}\hat{\psi}_-^\alpha = 0$ where $\hat{\psi}_-^\alpha = e^{\frac{1}{2}H^{1/2}}\psi_-^\alpha$ and one again concludes that $\psi_-^\alpha = 0$. Thus the solitons do not form enhanced representations of the Lorentz group.

It is useful to note that, in terms of the group element k defined by $A_+ = k\partial_+ k^{-1}$, one has

$$k^{-1} = Pexp\left(\sum_I K_I(\vec{x})\int_0^{x^+}\nu_+^I(y^+)dy^+\right) \ . \tag{5.3.28}$$

Furthermore observe that $K_I(\vec{x}_J) = \delta_{IJ}$ and hence

$$A_+(\vec{x}_I) = k(\vec{x}_I)\partial_+ k^{-1}(\vec{x}_I) = \nu_+^I(x^+) \ . \tag{5.3.29}$$

Thus although the gauge fields are spread-out over the whole of the multi-centred Taub-NUT space there is a sense in which the chiral mode ν_+^I is associated to the I-th pole in H. Furthermore far from the poles the field strength falls-off as $1/|\vec{x}|^2$ as expected for a massless charged particle in $4+1$ dimensions. However it is amusing to observe that near a pole $\vec{x}^I$ the gauge field

$$A_+ \sim \frac{RN_I/2}{RN_I/2 + |\vec{x}-\vec{x}_I|}\nu_+^I(x^+) \ , \tag{5.3.30}$$

is finite [173]. In particular for $|\vec{x}-\vec{x}_I| >> R$ the solution can be written terms of an infinite expansion of perturbative $g^2 = 4\pi^2 R$ corrections to the familiar $g^2/4\pi^2|\vec{x}-\vec{x}_I|$ Coloumb potential.

The energy-momentum tensor, $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}$, is readily found to be

$$\begin{aligned} T_{\mu\nu} \ &= \ \frac{1}{8\pi^2 R}\mathrm{tr}\Big[2\sqrt{H}D_\mu\phi_{\alpha\beta}D_\nu\phi^{\alpha\beta} + \frac{1}{2H^{3/2}}\partial_\mu H\partial_\nu H\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\sqrt{H}F_{\mu\rho}F_\nu{}^\rho \\ &\qquad -g_{\mu\nu}\Big(\sqrt{H}D_\rho\phi_{\alpha\beta}D^\rho\phi^{\alpha\beta} + \frac{1}{4}\frac{1}{H^{5/2}}\partial_i H\partial_i H\phi_{\alpha\beta}\phi^{\alpha\beta} + \frac{\sqrt{H}}{2}F_{\rho\sigma}F^{\rho\sigma} \\ &\qquad - \sqrt{H}[\phi^{\alpha\beta},\phi_{\beta\lambda}][\phi^{\lambda\rho},\phi_{\rho\alpha}]\Big)\Big] \ . \end{aligned} \tag{5.3.31}$$

So that on the discovered solution

$$\begin{aligned}
T_{++} &= \frac{1}{4\pi^2 R}\frac{1}{\sqrt{H}}\left(D_+\phi_{0\alpha\beta}D_+\phi_0^{\alpha\beta}+\sum_{IJ}\partial_i K_I\partial_i K_J\nu^I(x^+)\nu^J(x^+)\right)\\
T_{+-} &= -\frac{1}{32\pi^2 R}\frac{1}{H^{7/2}}\partial_i H\partial_i H\phi_{0\alpha\beta}\phi_0^{\alpha\beta}\\
T_{i+} &= -\frac{1}{8\pi^2 R}\frac{1}{H^{3/2}}\partial_i H\phi_{0\alpha\beta}D_+\phi_0^{\alpha\beta}\ .
\end{aligned}\tag{5.3.32}$$

Finiteness of the energy-momentum tensor implies that $D_+\phi^{\alpha}_{0\,\beta}=0$. This is satisfied easily by demanding $\phi^{\alpha}_{0\,\beta}$ be a constant, in particular I choose $\phi^{\alpha}_{0\,\beta}=0$ so that the unbroken gauge algebra is $\mathfrak{u}(N)$. With this extra step the energy momentum tensor again reduces to a very simple form where only T_{++} is non-zero and is given by

$$T_{++}=\frac{1}{4\pi^2 R}\frac{1}{\sqrt{H}}\mathrm{tr}\sum_{IJ}\partial_i K_I\partial_i K_J\nu^I(x^+)\nu^J(x^+)\ . \tag{5.3.33}$$

I then proceed to explicitly compute the integral over the internal $\mathbb{R}^3$ to find

$$\begin{aligned}
\mathcal{P}_+ &= \int d^3x dx^+\sqrt{-g}\,T_{++}\\
&= \frac{1}{4\pi}\sum_I N_I\mathrm{tr}\int dx^+\nu^I(x^+)\nu^I(x^+)\ .
\end{aligned}\tag{5.3.34}$$

This agrees with the abelian case above. Furthermore one sees that (5.3.34) corresponds precisely to n copies, where n is the number of centres of $\mathcal{M}_{mTN}$, of a WZWN model each at level N_I. However given that the value of N_I can be different for each I, one can't simply use a standard WZWN model on a three-manifold with n boundaries. I will return to this issue in the next section.

Next, consider at the gauge charges. For the first form of the action (5.3.2) I find

$$\begin{aligned}
J^\sigma(\Lambda) &= \frac{1}{8\pi^2 R}\mathrm{tr}\left[-2\sqrt{-g}\sqrt{H}F^{\sigma\lambda}D_\lambda\Lambda+\varepsilon^{\mu\nu\rho\sigma\lambda}\theta_\mu F_{\nu\rho}D_\lambda\Lambda\right]\\
&= \frac{1}{4\pi^2 R}\partial_\lambda\mathrm{tr}\left(\sqrt{-g}\sqrt{H}F^{\lambda\sigma}\Lambda+\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma\lambda}\theta_\mu F_{\nu\rho}\Lambda\right)\\
&\quad -\frac{1}{4\pi^2 R}\mathrm{tr}\left(\sqrt{-g}D_\lambda\left(\sqrt{H}F^{\lambda\sigma}\right)+\frac{1}{4}\varepsilon^{\mu\nu\rho\lambda\sigma}\mathcal{F}_{\mu\nu}F_{\rho\lambda}\right)\ ,
\end{aligned}\tag{5.3.35}$$

where the last line vanishes on-shell. The associated charges are

$$\begin{aligned}\mathcal{Q}(\Lambda(\infty)) &= \frac{1}{4\pi^2 R}\mathrm{tr}\sum_I \oint d\Omega_2 dx^+ \left[H\partial_r K_I + \varepsilon^{rjk}\theta_j\partial_k K_I\right]\nu_+^I\Lambda(\infty) \\ &= -\frac{1}{2\pi}\mathrm{tr}\sum_I N_I \int dx^+ \, \nu_+^I\Lambda(\infty) \ ,\end{aligned} \tag{5.3.36}$$

where $\Lambda(\infty)$ is any element of the unbroken gauge algebra. These charges only receive contributions from infinity and as such do not depend on the choice of θ. One sees that they are the natural non-abelian extension of (5.2.18) with the identification $\Lambda = 4\pi^2 R\Lambda_5$.

5.4 Gauge Symmetries and a WZWN-like Action

As mentioned above there are two choices for the five-dimensional action. The results in the previous section correspond to the first choice (5.3.2). In this section I explore some physical consequences of the other choice of the action (5.3.4). I will then use this analysis to motivate a WZWN-like model as the effective action for the chiral soliton modes found above.

5.4.1 Physical 'Gauge' Transformations

The main difference between the two-forms for the action can be seen from their gauge symmetry. While the first form is gauge invariant the second is not. In particular the second form of the action (5.3.4) transforms as (assuming boundary conditions that allow us to ignore boundary terms in x^+)

$$\begin{aligned}\delta_\Lambda S &= -\frac{1}{4\pi}N_{D6}\int d^2x \mathrm{tr}\left((\partial_+ A_-(\infty) - \partial_- A_+(\infty))\Lambda(\infty)\right) \\ &\quad + \frac{1}{4\pi}\sum_I N_I \int d^2x \mathrm{tr}\left((\partial_+ A_-(\vec{x}_I) - \partial_- A_+(\vec{x}_I))\Lambda(\vec{x}_I)\right)\end{aligned} \tag{5.4.1}$$

One can make the first line vanish by imposing a suitable boundary condition at infinity. However for the other terms it seems more natural to restrict the gauge symmetry so that

$$\Lambda(\vec{x}_I) = 0 \ . \tag{5.4.2}$$

As will be seen this has the effect of introducing additional degrees of freedom that live at the poles $\vec{x}_I$. These arise because there are now transformations of the soliton solution generated by $\Lambda(\vec{x}_I)$ which lead to physically distinct states.

To continue I evaluate the action (5.3.4) on the full space of BPS solutions, including dependence of g on $x^+, x^-, \vec{x}$. The first term of the action is still vanishing. However substituting the general ansatz (5.3.15)-(5.3.17) into the second form of the action (5.3.4) I find

$$\begin{aligned} S_{BPS} &= \frac{1}{8\pi^2 R}\mathrm{tr}\int \mathcal{F}\wedge(A\wedge dA + \frac{2}{3}A\wedge A\wedge A) \\ &= \frac{1}{8\pi^2 R}\int \partial_i H(CS)_{+-i}dx^+dx^-d^3x \ . \end{aligned} \tag{5.4.3}$$

Evaluating the action on our BPS sector gives

$$S_{BPS} = \frac{1}{8\pi^2 R}\mathrm{tr}\int \partial_i H(\partial_i(A_+A_-) + A_i\partial_+A_- - A_-\partial_+A_i)dx^+dx^-d^3x \ , \tag{5.4.4}$$

where I have used the fact that $F_{i-}=0$ and assumed boundary conditions along x^- that allow one to drop boundary terms in x^-.

There are two ways to proceed. The first is analogous to the classic construction of [123]. In that treatment one integrates over the A_+ gauge field which imposes the constraint $F_{-i}=0$. Here I do not integrate over A_+. Rather I have imposed the BPS conditions, which includes the constraint $F_{-i}=0$, and evaluated the action. To this end I integrate the first term in (5.4.4) by parts and, observing that

$$\partial_i\partial_i H = -2\pi\sum N_I R\delta^3(\vec{x}-\vec{x}_I) \ , \tag{5.4.5}$$

I find a contribution

$$S_{BPS} = \frac{1}{4\pi}\sum_I N_I \mathrm{tr}\int dx^+dx^-A_+(\vec{x}_I)A_-(\vec{x}_I) + \dots \ . \tag{5.4.6}$$

To continue in analogy with [123] I assume a boundary condition such that $A_+(x_I) =$

0 for each I. With this condition the full action reduces to:

$$\begin{aligned} S_{BPS} &= -\sum_I \frac{N_I}{4\pi}\mathrm{tr}\int dx^+dx^-\, g(\vec{x}_I)\partial_+ g^{-1}(\vec{x}_I)g(\vec{x}_I)\partial_- g^{-1}(\vec{x}_I) \\ &\quad +\frac{1}{8\pi^2 R}\mathrm{tr}\int d^5x\,\partial_i H[g^{-1}\partial_- g, g^{-1}\partial_+ g]\, g^{-1}\partial_i g\ . \end{aligned} \tag{5.4.7}$$

This is essentially a WZWN model with n two-dimensional 'boundaries' located at the poles of H each with level N_I (although we recall that only $N_I = 1$ corresponds to a completely smooth multi-centred Taub-NUT space). The difference with a traditional WZWN model is that in this case the topological term is five-dimensional and the two-dimensional 'boundary' contributions arise from the poles of H. Nevertheless it plays the same role as the familiar three-dimensional term. In particular the associated equation of motion is restricted to the poles and is given by

$$\partial_+(g(\vec{x}_I)\partial_- g^{-1}(\vec{x}_I)) = 0\ , \tag{5.4.8}$$

for each I. I thus obtain a theory of n independent two-dimensional group-valued fields $g(\vec{x}_I)$. The solution to this is simply

$$g(\vec{x}_I) = \ell_I(x^-)r_I(x^+)\ . \tag{5.4.9}$$

for arbitrary group elements $\ell_I(x^-)$ and $r_I(x^+)$. However one must ensure that the boundary condition $A_+(x_I) = 0$ is satisfied. One finds that this implies

$$r_I = k^{-1}(\vec{x}_I)\ , \tag{5.4.10}$$

and hence

$$g(\vec{x}_I) = \ell_I(x^-)k^{-1}(\vec{x}_I)\ . \tag{5.4.11}$$

Thus one is left with a single independent group element $\ell_I(x^-)$ in addition to the original solution $k^{-1}(\vec{x}_I)$

The second approach is to include the 'boundary' term (5.4.6) into the action which I again evaluate on the BPS solutions, *i.e.* I do not impose any conditions on

A_+ at the poles. In this case I find

$$\begin{aligned} S_{BPS} &= \sum_I \frac{N_I}{4\pi}\mathrm{tr}\int dx^+dx^-\, k(\vec{x}_I)\partial_+k^{-1}(\vec{x}_I)\partial_-g^{-1}(\vec{x}_I)g(\vec{x}_I) \\ &\quad +\frac{1}{8\pi^2 R}\mathrm{tr}\int d^5x\,\partial_i H[g^{-1}\partial_-g, g^{-1}\partial_+g]\,g^{-1}\partial_i g\ . \end{aligned} \tag{5.4.12}$$

Here the standard quadratic kinetic term for g has been removed and replaced by a linear term coupled to the background field k. The associated equation of motion still only receives contributions from the poles but has a less familiar form:

$$\begin{aligned} 0 &= \partial_- g k(\vec{x}_I)\partial_+k^{-1}(\vec{x}_I)g^{-1} + gk(\vec{x}_I)\partial_+k^{-1}(\vec{x}_I)\partial_-g^{-1} \\ &\quad +g\partial_+g^{-1}g\partial_-g^{-1} - g\partial_-g^{-1}g\partial_+g^{-1}\ , \end{aligned} \tag{5.4.13}$$

for each I. To solve this one can write

$$g(\vec{x}_I) = \ell_I(x^+, x^-)k^{-1}(\vec{x}_I)\ , \tag{5.4.14}$$

for some ℓ_I that is now allowed to depend on both x^- and x^+. Substituting this into (5.4.13) I simply find, for each I,

$$\left[\ell_I\partial_+\ell_I^{-1}, \ell_I\partial_-\ell_I^{-1}\right] = 0\ . \tag{5.4.15}$$

There are essentially two ways to satisfy this equation. Firstly, if $\ell_I\partial_-\ell_I^{-1} = 0$ then one has $\ell_I = \ell_I(x^+)$. This means that $g = \ell_I k^{-1}(\vec{x}_I)$ is a function only of x^+ and hence ℓ_I can be absorbed into a redefinition of $\nu_I(x^+)$. The second solution is to demand $\ell_I\partial_+\ell_I^{-1} = 0$ so one has $\ell_I = \ell_I(x^-)$. In this case I recover the same solutions seen above by imposing the vanishing of $A_+(\vec{x}_I)$.

In summary I find that with the second choice of action (5.3.4) there are some gauge modes which are physical. In particular I find that the solution space includes the modes $\ell_I(x^-)$ that arises from the broken gauge modes. Hence one can think of it as a physical Goldstone mode and the WZWN-like model as its low energy effective action. However I do not expect such modes to arise from the D-brane analysis and hence I conclude that (5.3.4) is the wrong choice of action.

5.4.2 An Action for the Soliton Modes

I now return to the original action (5.3.2). Here I can simply adapt the argument above. It has been shown that the D4-D6 strings can be realised in the non-Abelian theory as solitons. I have evaluated their energy and momentum and shown that they agree with that of a chiral half of a WZWN model. To capture the effective dynamics of these solitons I therefore propose that the action (5.4.7) can be used with a slightly modified interpretation. In particular, recall that the solution to the equations of motion can be written as

$$g(\vec{x}_I) = \ell_I(x^-) r_I(x^+) \ , \tag{5.4.16}$$

for arbitrary left and right moving modes ℓ_I and r_I. To make contact with the discovered solitons, first set ℓ_I to the identity and identify

$$\nu_+^I(x^+) = r_I(x^+) \partial_+ r_I^{-1}(x^+) \ . \tag{5.4.17}$$

i.e. $r_I(x^+) = k(\vec{x}_I)$ in (5.3.29). One also sees that taking a non-trivial $\ell_I(x^-)$ can be viewed as performing the gauge transformation: $A_+(\vec{x}_I) = \ell k \partial_+(k^{-1}\ell^{-1})$ and $A_-(\vec{x}_I) = \ell \partial_- \ell^{-1}$. Therefore I consider the other chiral half to be pure gauge and simply discard it. This is consistent with the discussion above where such gauge modes were physical and therefore not discarded.

5.5 Summary and Comments

In this chapter I have studied how the charged D4-D6 strings which arise from a D4-brane intersecting with a D6-brane are realised in the M5-brane worldvolume theory. In particular it was shown that there are smooth soliton solutions of the five-dimensional Yang-Mills gauge theory arising from the M5-brane reduced on the circle fibration of multi-centred Taub-NUT space that have the right charges to be identified with the D4-D6 strings. I also considered the physical consequences of the two choices of action and how the second choice leads to additional physical soliton zero-modes which do not match the string theory analysis. Lastly I obtained a WZWN-like model for the solitons but where the topological term is five-dimensional. I thus conclude that five-dimensional maximally supersymmetric Yang-Mills contains the charged states predicted from the D-brane construction, albeit as solitons.

I briefly mention some bulk eleven-dimensional aspects of our solutions. The states which have been identified arise as stretched D4-D6-strings. In the string theory picture these states are localized to the intersection. In M-theory they lift to M2-branes that wrap the M-theory circle. Since the M-theory circle shrinks to zero at the poles of H the M2-brane worldvolume theory develops a potential $V \propto H^{-1/2}$ and so the energy is minimized by sticking to the poles $\vec{x}_I$, in agreement with the microscopic string theory picture.

The solutions presented here are given in terms of harmonic forms which can also be associated to the existence of non-trivial two-cycles in multi-centred Taub-NUT. These two-cycles are caused by the shrinking of the circle fibration at the poles of H and so can be thought of as connecting two distinct poles. M2-branes wrapping these cycles are in bi-fundamental representations of $U(1) \times U(1)$ subgroups of a $U(1)^{N_{D6}-1}$ gauge group whose bulk gauge field arises from a Kaluza-Klein reduction of the M-theory three-form $C \sim \sum C_I \wedge \omega_I$ (here one is thinking of multi-centred Taub-NUT as compact, or replacing it by a similar compact space). When all the D6-branes coalesce this group is enhanced and the wrapped M2-branes provide the additional gauge bosons to form the adjoint of $SU(N_{D6})$. However the states found here are different. One reason is simply that for single centred Taub-NUT there is a harmonic two-form but no non-trivial two-cycle. More generally one sees that the soliton profile is $A_+ \sim \sum K_I \nu_+^I$ and $0 \leq K_I \leq 1$ with $K_I = 1$ *iff* $\vec{x} = \vec{x}_I$. Thus the I-th soliton is peaked at the I-th pole and furthermore vanishes at all the other poles. This means that the states found here do not correspond to M2-branes which are wrapped on the non-trivial two-cycles. Rather these states are trapped at the poles, as discussed above. As such they are naturally associated to fundamental representations of the bulk enhanced gauge group, providing charged states of the bulk $SU(N_{D6})$ gauge group. From the point of view of the M2-brane worldvolume theory the wrapped M2-brane states arise as kink-like solitons, interpolating between pairs of poles as in [181].

Chapter 6

Conclusions

In this thesis I have presented a discussion of the history, background, and modern understanding of M-theory. This work cannot hope to be comprehensive, the field has expanded exponentially from its earliest days and at this time one is as likely to meet pure mathematicians and condensed matter physicists at a String Theory conference as those with a simple interest in analysing M-theory. I hope however to have provided a coherent narrative which outlines the necessary background and conceptual understanding of the work which I have undertaken as a part of my studies. I hope also to have presented this work clearly and intelligibly. I now provide a summary of the results and themes contained in this thesis.

In Chapter 2 I provided a discussion on the BLG theory description of M2-branes. I showed how the theory arises, how it is formulated, and justified its interpretation as describing a pair of M2-branes. I also explained how its $N = 8$ supersymmetry requirement is ultimately too restrictive for the model to provide a generalised description of multiple M2-branes. The ABJM model was briefly introduced and I described how, by reducing the supersymmetry to $N = 6$, this new $U(N) \times U(N)$ with level k model can describe N M2-branes on a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold. I also clarified that the BLG theory is contained within the ABJM theory as the limit where $N = 2$ and $k = 1$. From this point I discussed M5-branes and our current difficulties with their description. In particular the supposed worldvolume six-dimensional $(2, 0)$ theory is not well understood and it is believed that no manifestly Lorentz invariant lagrangian exists for the theory. Finally I spent some time discussing non-abelian extensions to the $(2, 0)$ theory. The hope with such models is that they could provide insights into novel brane configurations which may elucidate features of the $(2,0)$ theory which we do not as yet understand. They also provide an interesting arena for generating new systems of M2 and M5-branes for study.

In Chapter 3, I discussed the case of an extended system of the $(2,0)$ theory in which the non-dynamical three-form from the extension is turned on in a null direction. The resulting system was found to have an $SO(2) \times SO(2) \times SO(6)$ symmetry and led to a vacuum configuration with dynamics consisting of quantum mechanics acting on the Hitchin moduli space. It was argued that this system has a number of different brane interpretations. In one sense it can be viewed as two M5-branes compactified on $\mathbb{T}^2 \times S^1_-$ with S^1_- denoting a null circle with a null momentum fixed by the topological charge. The other interpretation of this system was argued to be a pair of null M2-branes on $\mathbb{R}_+ \times \mathbb{R}^2$—branes which have had a light-like boost along a transverse direction to both branes. These interpretations were argued to be related to one another using various string and M-theory dualities.

In Chapter 4, I extended the analysis of this system and presented an action for the system. This is an interesting result as the action in question is a maximally supersymmetric non-Lorentzian $(2+1)$-dimensional field theory. A similar construction for the case of the theory in [146] was also presented. In this case a corresponding maximally supersymmetric non-Lorentzian $(4+1)$-dimensional field theory was found. For both these actions the BPS moduli space dynamics which are expected to classify the system are enforced through the presence of Lagrange multiplier fields. It would be interesting to derive these actions by taking a non-Lorentzian scaling limit, perhaps something like a mixture of Carrollian and Galilean limits in the sense of [182], directly within the parent Lorentzian field theory without embedding the branes into eleven dimensions. Indeed one may expect that many supersymmetric field theories admit non-Lorentzian limits of this type which preserve all the supersymmetries and whose on-shell dynamics reduce to motion on a moduli space. Such a limit makes the Manton approximation, where the dynamics are described by slow motion on a soliton moduli space, exact. It also raises the question of what is the classification of all field theories with 16 supersymmetries if one does not impose the condition of Lorentz invariance. It would also be interesting to see whether one could obtain the M2-action of this chapter from ABJM; such a construction should be possible and it would be interesting to shed light on the connection between these descriptions.

Finally, in Chapter 5 I described how the chiral modes found in string theory along the D4-D6 brane intersection are realised from the M-theory perspective. I provided a discussion of this both in the abelian (single M5-brane case) and non-abelian (multiple M5-brane case) systems. I argued that from the M-theory perspective the chiral states arise as soliton modes—thus evading Goldstones theorem—and demonstrated a way for such states to arise from the M5-brane. I recalled that the system of N stacked D4-branes intersecting separated D6-branes has an M-theory equivalent of N M5-branes wrapped on the multi-Taub-NUT space. I then argued that after reduction on the Taub-NUT fibration, and the construction of a non-abelian generalisation of the resulting theory, BPS states can be found which naturally correspond to the chiral modes sought. These were shown to be manifested as a WZWN like model with a five-dimensional topological term. The chiral modes were shown to have the correct properties for identification with the corresponding string theory D4-D6 strings.

M-theory has proven time and time again to be a subject which will not reveal its secrets easily. There are many outstanding problems and results which are, as yet, simply not understood. It may be that in 20 years time the subject is completely changed after new hints push it in unanticipated directions. Throughout though, many more researchers will toil long and hard to keep travelling along this road; to witness its surprising twists and turns in the hope that one day it will answer that age old question raised at the outset. This thesis, I hope, constitutes a small part of this journey.

Appendix A

Conventions in the Text

Throughout the work I use the mostly positive metric

$$\eta = (- + \ldots +) \tag{A.0.1}$$

In addition the totally anti-symmetric tensor is defined to be

$$\varepsilon^{01\ldots(d-1)} = +1 \tag{A.0.2}$$

The hodge dual for p-forms in D dimensions is defined as

$$(\star F)_{\mu_1\mu_2\ldots\mu_{D-p}} = \frac{1}{p!}\sqrt{-g}\,\varepsilon_{\mu_1\mu_2\ldots\mu_{D-p}}{}^{\nu_1\nu_2\ldots\nu_p}F_{\nu_1\nu_2\ldots\nu_p} \tag{A.0.3}$$

Unless specified, the following lightcone coordinates will be used

$$x^+ = \frac{x^5 + x^0}{\sqrt{2}} \qquad x^- = \frac{x^5 - x^0}{\sqrt{2}}\ . \tag{A.0.4}$$

In these coordinates

$$\begin{aligned} \eta_{+-} &= \eta_{-+} = 1 \\ \varepsilon_{1234+-} &= \varepsilon_{+-1234} = -1\ . \end{aligned} \tag{A.0.5}$$

The following is of primary relevance for Chapter 3 but similar conventions are used elsewhere where appropriate. For spinors it will be useful to introduce the following conventions:

$$\begin{aligned} \Gamma_\pm &= \frac{\Gamma_5 \pm \Gamma_0}{\sqrt{2}} \\ \Gamma_{05} &= \Gamma_{+-} \ . \end{aligned} \tag{A.0.6}$$

So that

$$\begin{aligned} \Gamma_-\chi = \Gamma_-\chi_+ &= -\sqrt{2}\Gamma_0\chi_+ \\ \Gamma_+\chi = \Gamma_+\chi_- &= \sqrt{2}\Gamma_0\chi_- \\ \Gamma_\pm\chi_\pm &= 0 \\ \Gamma_-\Gamma_+\chi &= 2\chi_- \\ \Gamma_+\Gamma_-\chi &= 2\chi_+ \ . \end{aligned} \tag{A.0.7}$$

Complex coordinates are introduces as

$$z = x^1 + ix^2 \ , \tag{A.0.8}$$

so that

$$\begin{aligned} g_{z\bar{z}} &= \frac{1}{2} \qquad \varepsilon_{-+z\bar{z}34} = \frac{i}{2} \\ D \equiv D_z &= \frac{1}{2}\left(D_1 - iD_2\right) \qquad \bar{D} \equiv D_{\bar{z}} = \frac{1}{2}\left(D_1 + iD_2\right) \ . \end{aligned} \tag{A.0.9}$$

I also define

$$\begin{aligned} \hat{\Gamma}_z &= \frac{1}{2}(\hat{\Gamma}_1 - i\hat{\Gamma}_2) = \frac{1}{2}(\Gamma_{01} - i\Gamma_{02}) \\ \hat{\Gamma}_{\bar{z}} &= \frac{1}{2}(\hat{\Gamma}_1 + i\hat{\Gamma}_2) = \frac{1}{2}(\Gamma_{01} + i\Gamma_{02}) \ . \end{aligned} \tag{A.0.10}$$

There is also a complex scalar introduced as

$$Z = Y^4 + iY^3 \ , \tag{A.0.11}$$

which implies the following gamma matrix definitions

$$\begin{aligned}\hat{\Gamma}_Z &= \frac{1}{2}(\hat{\Gamma}_3 - i\hat{\Gamma}_4) = \frac{1}{2}(\Gamma_{054} - i\Gamma_{053}) \\ \hat{\Gamma}_{\bar{Z}} &= \frac{1}{2}(\hat{\Gamma}_3 + i\hat{\Gamma}_4) = \frac{1}{2}(\Gamma_{054} + i\Gamma_{053}) \ .\end{aligned} \tag{A.0.12}$$

Appendix B

Useful Fierz Identities

Here I list some Fierz identitities which prove useful in proving supersymmetry of the actions under discussion in Chapter 4. In Section 4.2 I utilise the following:

$$
\begin{aligned}
0 &= \langle\Psi_+^T, [X^I, (\epsilon_-^T\hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^I\Psi_+), \Psi_+]\rangle + \langle\Psi_+^T, [X^I, (\epsilon_-^T\hat{\Gamma}^J\Psi_+), \hat{\Gamma}_{Z\bar{Z}}\hat{\Gamma}^{IJ}\Psi_+]\rangle \\
0 &= \langle\Psi_+^T, [X^I, (\epsilon_+^T\hat{\Gamma}^I\hat{\Gamma}_{\bar{z}}\Psi_+), \hat{\Gamma}_z\Psi_-]\rangle - \langle\Psi_+^T, [X^I, (\epsilon_+^T\hat{\Gamma}^I\hat{\Gamma}_z\Psi_+), \hat{\Gamma}_{\bar{z}}\Psi_-]\rangle \\
&\quad + \langle\Psi_+^T, [X^I, (\epsilon_+^T\hat{\Gamma}^I\hat{\Gamma}_Z\Psi_+), \hat{\Gamma}^I\hat{\Gamma}_{\bar{Z}}\Psi_-]\rangle - \langle\Psi_+^T, [X^I, (\epsilon_+^T\hat{\Gamma}^I\hat{\Gamma}_{\bar{Z}}\Psi_+), \hat{\Gamma}^I\hat{\Gamma}_Z\Psi_-]\rangle \\
0 &= \langle\Psi_+^T, [Z, (\epsilon_-^T\hat{\Gamma}_Z\Psi_-), \Psi_+]\rangle + 2\langle\Psi_+^T, [Z, (\epsilon_-^T\hat{\Gamma}_{\bar{z}Z}\Psi_+), \hat{\Gamma}_z\Psi_-]\rangle \\
&\quad - \langle\Psi_+^T, [Z, (\epsilon_-^T\hat{\Gamma}^I\Psi_+), \hat{\Gamma}^I\hat{\Gamma}_Z\Psi_-]\rangle \\
0 &= \langle\Psi_-^T, [Z, (\epsilon_+^T\hat{\Gamma}_Z\Psi_+), \Psi_-]\rangle + 2\langle\Psi_-^T, [Z, (\epsilon_+^T\hat{\Gamma}_{zZ}\Psi_-), \hat{\Gamma}_{\bar{z}}\Psi_+]\rangle \\
&\quad - \langle\Psi_+^T, [Z, (\epsilon_+^T\hat{\Gamma}^I\Psi_-), \hat{\Gamma}^I\hat{\Gamma}_Z\Psi_+]\rangle \ . \qquad \text{(B.0.1)}
\end{aligned}
$$

There are also similar identities where $Z \to \bar{Z}$. These can be derived from the vanishing of the cubic fermion terms that arise in δS for the maximally supersymmetric M2-brane theory and then splitting-up the fields into their various components, *e.g.* $\Psi = \Psi_+ + \Psi_-$, $\epsilon = \epsilon_+ + \epsilon_-$, $X^I \to X^I, Z, \bar{Z}$, where the sign indicates their chirality with respect to $\hat{\Gamma}_{034}$.

In Section 4.3 the following Fierz identities arise:

$$
\begin{aligned}
0 &= \mathrm{tr}\left(\Psi_-^T[(\epsilon_-^T\Gamma_0\Psi_+), \Psi_-]\right) + \mathrm{tr}\left(\Psi_+^T[(\epsilon_-^T\Gamma_m\Psi_-), \Gamma_0\Gamma_m\Psi_+]\right) \\
&\quad + \mathrm{tr}\left(\Psi_-^T[(\epsilon_-^T\Gamma_m\Psi_-), \Gamma_0\Gamma_m\Psi_+]\right) + \mathrm{tr}\left(\Psi_-^T[(\epsilon_-^T\Gamma_0\Gamma_m\Psi_+), \Gamma_m\Psi_-]\right) \\
0 &= \mathrm{tr}\left(\Psi_-^T[(\epsilon_+^T\Gamma_0\Psi_-), \Psi_-]\right) - \mathrm{tr}\left(\Psi_-^T[(\epsilon_+^T\Gamma_0\Gamma_m\Psi_-), \Gamma_m\Psi_-]\right) \ , \qquad \text{(B.0.2)}
\end{aligned}
$$

where $m = 1, 2, 3, 4, 6, 7..., 10$ (*i.e.* $m \neq 5$). These can be derived from the vanishing of the cubic fermion terms that arise in δS in five-dimensional maximally supersymmetric Yang-Mills theory and then splitting-up the fields into their various components, *e.g.* $\Psi = \Psi_+ + \Psi_-$, $\epsilon = \epsilon_+ + \epsilon_-$ where the sign indicates their chirality with respect to Γ_{05}.

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